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Abstract: We propose a novel strategy-proof dynamic auction for efficiently allocating heterogeneous indivisible commodities. The auction applies to all unimodular demand types of Baldwin and Klemperer's necessary and sufficient condition for the existence of competitive equilibrium which accommodate a variety of complements, substitutes, gross substitutes and complements, and any other kinds. Although bidders are not assumed to be price-takers so they can act strategically, this auction induces bidders to bid truthfully, yielding efficient outcomes. Sincere bidding is shown to be an ex post perfect Nash equilibrium of the auction. The trading rules are simple, detail-free, privacy-preserving, error-tolerant, and independent of any probability distribution assumption.

Keywords: Dynamic Auction Design, Equilibrium, Incentive Compatibility, Unimodular Demand Types, Indivisibility, Incomplete Information.
MSC: 91B26, 91A27, 91B50, 91A10.

1 Introduction

This paper offers a general, efficient, and strategy-proof dynamic design for auctioning a wide variety of heterogeneous indivisible commodities/items to many bidders. Every bidder has a private valuation on every of his interested bundles of items, may demand any number of items and act strategically rather than truthfully. A consequence of our design resolves an important issue concerning complements raised by Milgrom (2017, p.45), who says: "Markets for complements can be much harder than markets for substitutes and can require greater planning and coordination."[1]

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[1] This echoes Milgrom (2000, 2004), Jehiel and Moldovanu (2003), Noussair (2003), Porter et al. (2003), and Maskin (2005) on the same issue.

In dynamic auction design, prices play an instrumental rule in guiding the market toward a competitive equilibrium. In this paper, we use the standard notion of competitive equilibrium. The pricing rule is anonymous and linear for all agents. This rule is common, easy, and practical. Our dynamic auction design applies to all unimodular demand types of Baldwin and Klemperer (2019) which are *a necessary and sufficient condition for the existence of competitive equilibrium* regardless of whether the commodities are complements, (gross) substitutes (GS), gross substitutes and complements (GSC), or any other kinds. Unimodular demand types are rich, unify existing sufficient conditions[2] and can also identify previously-unknown environments in which a competitive equilibrium still exists. Baldwin and Klemperer (2014, 2019) have also shown that there are far more classes of complements than of substitutes for equilibrium existence.

While Baldwin and Klemperer (2019) have established this important equilibrium existence theorem via a nonconstructive method in an equilibrium model with price-taking agents and complete information, this article addresses a related, but distinct, fundamental problem of how competitive equilibrium prices can be formed and found and efficient allocations can be identified in an incomplete information environment with strategic bidders. We propose a dynamic auction and show that sincere bidding is an ex post perfect Nash equilibrium of the auction game of incomplete information and the auction yields a competitive outcome. Our auction is the first efficient and strategy-proof dynamic mechanism under the necessary and sufficient condition of Baldwin and Klemperer (2019).

Besides, we shall also highlight two other major contributions of this paper. First, a salient feature of our auction design is a new concept of "a search set," which makes our approach universal, not ad hoc. Our auction converges globally for every unimodular demand type from any starting point to a competitive equilibrium. Our approach is novel, combinatorial, general, employing only convexity and unimodularity. It goes beyond the conventional ones which use the familiar property of submodularity; see e.g., Gul and Stacchetti (2000) and Ausubel (2006). Submodularity indeed holds for (gross) substitutes but, in general, does not hold for other demand types. Our auction provides also an innovative algorithm for solving a class of general constrained discrete optimization problems

[2]Earlier existence results include Koopmans and Beckmann (1957), Shapley and Shubik (1971), Kelso and Crawford (1982), Gul and Stacchetti (1999), Danilov et al. (2001), Sun and Yang (2006), Crawford (2008), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Shioura and Yang (2015).

in which functions are not given explicitly. This is in marked contrast to the literature in which functions are given explicitly and algorithms work directly on the functions; see e.g., Murota (2003), Fujishige (2005), Lee and Leyffer (2012). Furthermore, we prove that the set of competitive equilibrium price vectors in our market exhibits a striking geometric structure being an integral polytope, sharpening and extending the lattice results obtained by Shapley and Shubik (1971), Gul and Stacchetti (1999) and Ausubel (2006) for substitutes. A lattice is not necessarily a polytope.

Second, it is well-recognized that *strategy-proof dynamic mechanisms* have important advantages over *strategy-proof direct/static mechanisms* in their capacity of alleviating bidders' concern about privacy and reducing computational complexity, payoff uncertainty and information cost; see e.g., Rothkopf et al. (1990), McMillan (1994), Ausubel (2004, 2006), Ausubel and Milgrom (2005), Perry and Reny (2005), Bergemann and Morris (2007), Rothkopf (2007), and Milgrom (2007, 2017). Besides, possessing such desirable properties, the current auction can tolerate various dishonest behaviors and mistakes made by bidders and allow them to learn, adjust, and correct. Unlike the conventional approach of a huge penalty for violation, we adopt a lenient policy and show that no bidder will end up with a negative payoff as long as he can differentiate a positive number from a negative one, no matter how his competitors bid. The current auction is independent of any probability distribution assumption, detail-free, and robust against any regret and needs only a minimal common knowledge assumption that the unimodular demand type of commodities is known. This is desirable and important; see Wilson (1987).

The rest of this article goes as follows. The auction model is introduced in Section 2. The structure of the set of competitive equilibria and other properties of the model are explored in Section 3. The basic dynamic auction design and convergence are discussed in Section 4. The strategy-proof dynamic auction built upon the basic dynamic auction and its strategic properties are examined in Section 5.

1.1 A Brief Literature Review

Most dynamic auctions were designed for (gross) substitutes—the benchmark condition introduced by Kelso and Crawford (1982). These include Crawford and Knoer (1981), Kelso and Crawford (1982), Demange et al. (1986), Gul and Stacchetti (2000), Milgrom (2000), Ausubel (2004, 2006), Hatfield and Milgrom (2005), Milgrom and Strulovici (2009), and

Murota et al. (2016), etc. Among these, sincere bidding is an ex post Nash equilibrium for the assignment market (see e.g., Demange et al. 1986 and Andersson and Svensson 2016), and an ex post perfect Nash equilibrium for those of Ausubel (2004, 2006). In contrast, there are only very few results concerning complements; see Sun and Yang (2009) for a dynamic auction for gross substitutes and complements and Candogan et al. (2015) for an iterative auction for tree valuations exhibiting substitutes and complements. These papers and the current paper all use anonymous and linear pricing rules. Sun and Yang (2014) proposed a strategy-proof dynamic auction for multiple complements using anonymous and nonlinear pricing. Furthermore, discriminatory and nonlinear pricing rules are applied to package auctions; see Ausubel and Milgrom (2002), Mishra and Parkes (2007), and De Vries et al. (2007) for ascending auctions. These pricing rules are so general that they can charge people differently for the same bundle of goods and offer solutions for markets lacking competitive equilibrium. However, anonymous and linear pricing rules have distinct advantages over these rules so are more commonly used in both theory and practice.

In the traditional analyses, it has been essential to assume that agents are price-takers or have no market power at all (see Debreu and Scarf 1963, Aumann 1964, and Arrow and Hahn 1971). Unfortunately, this assumption can hardly be satisfied in any real life auction, as the number of bidders is usually small and bidders do possess considerable market power so it is inconceivable that they would not bid strategically if it were in their interests to do so. See Kojima and Pathak (2009) for a discussion on large markets. Our paper aims to provide an efficient and strategy-proof dynamic auction mechanism for general markets where no one has all information but every bidder possesses some private information and may act strategically. See Hayek (1945) and Hurwicz (1971) on such fundamental issues.

2 The Model

An auctioneer or a seller wants to sell a set $N = \{1, 2, \cdots, n\}$ of n indivisible items to a group B of m potential bidders. Some of the items can be heterogeneous and the other can be identical. Identical items will be labelled differently. This way of treating indivisible items in a competitive equilibrium model causes no loss of generality as identical units of the same commodity can be treated as different items but will have the same equilibrium price. Let $\mathbb{R}^N$ denote the n-dimensional Euclidean space, where each coordinate is indexed by a number from the set N. Let $\mathbb{Z}^N$ stand for the set of all integer vectors in $\mathbb{R}^N$. For every $i \in N$, let $e(i)$ denote the ith unit vector in $\mathbb{R}^N$. A subset S of N represents a bundle of items in S. For easy exposition, we regard a set S and the corresponding vector $\sum_{i\in S} e(i)$ as the same bundle.

Every bidder (he) $j \in B$ has a utility function $u^j : \{0,1\}^N \to \mathbb{Z} \cup \{-\infty\}$ specifying his valuation $u^j(x)$ (in units of money, say, in dollars) on every bundle x, where $\{0,1\}^N$ denotes the set of all bundles of items. The seller (she) denoted by 0 has a reserve price function $u^0 : \{0,1\}^N \to \mathbb{Z}\cup\{-\infty\}$. Let $B_0 = B\cup\{0\}$ stand for the set of all market participants (all bidders and the seller). In general, when we talk about a generic agent who can be a bidder or the seller, we treat the agent as female. Let $\mathrm{dom}(u^j) = \{x \in \{0,1\}^N \mid u^j(x) > -\infty\}$ denote the *effective domain* of u^j for every agent $j \in B_0$. A bundle x is *unacceptable* to an agent $j \in B_0$ if and only if $u^j(x) = -\infty$, i.e., $x \notin \mathrm{dom}(u^j)$.

All agents have quasi-linear utilities in money. That is, every agent j's utility over any bundle x and any amount c of money can be written as $U^j(x, c) = u^j(x) + c$ for $j \in B_0$. Every agent has a limited but enough amount of budget so that she does not face any budget constraint (*No Budget Constraint Condition*). Note that when a commodity is sold with a negative price, this means that the commodity can be bad and the seller will pay the price. So our model can accommodate indivisible goods as well as bads. We use $\mathcal{M} = (u^j, j \in B_0, N)$ or simply $\mathcal{M}$ to represent the market. A submarket is what is left in the market $\mathcal{M}$ by deleting a number of bidders and a number of items.

An *allocation* of items in N is a *redistribution* $X = (x^j, j \in B_0)$ of items among all market participants in B_0 such that $\sum_{j\in B_0} x^j = \sum_{i\in N} e(i)$ and $x^j \in \{0,1\}^N$ for all $j \in B_0$. At allocation X, agent $j \in B_0$ receives bundle x^j. An allocation $X = (x^j, j \in B_0)$ is *feasible* if $x^j \in \mathrm{dom}(u^j)$ for every agent $j \in B \cup \{0\}$. We assume that the market has at least one feasible allocation (*Feasibility Condition*). This is a general and minimal assumption, meaning that every item can be acceptable to some agents in some ways and every agent has at least one acceptable bundle.[3] An allocation $X = (x^j, j \in B_0)$ is

[3]The following market trivially satisfies this assumption. The set $\mathrm{dom}(u^j)$ of every bidder $j \in B$ contains at least one nonzero vector and also the dummy bundle $\mathbf{0}$ with $u^j(\mathbf{0}) = 0$. So every bidder has the option of buying nothing and is interested in buying some items. The set $\mathrm{dom}(u^0)$ of the seller equals $\{0,1\}^N$ with $u^0(\mathbf{0}) = 0$. This means that the seller will not sell but retain a bundle if the price of the

efficient if $\sum_{j\in B_0} u^j(x^j) \geq \sum_{j\in B_0} u^j(y^j)$ for every allocation $Y = (y^j, j \in B_0)$. Given an efficient allocation X, let $R(N) = \sum_{j\in B_0} u^j(x^j)$. We call $R(N)$ *the market value* of the items which is the same for all efficient allocations. Clearly, an efficient allocation must be feasible.

An n-vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^N$ specifies a price p_i for every item $i \in N$ and is the same for all bidders. This is an anonymous and linear pricing rule, which is easy and practical and has long and widely being used in theory and practice. Every bidder $j \in B$ maximizes his profit and his demand set $D^j(p)$ is given by

$$D^j(p) = \arg\max_{x\in\{0,1\}^N}\{u^j(x) - p\cdot x\}, \tag{1}$$

where $p \cdot x = \sum_{i\in N} p_i x_i$. At prices $p \in \mathbb{R}^N$, the seller chooses bundles to maximize her revenues and her demand set $D^0(p)$ is given by

$$\begin{aligned} D^0(p) &= \arg\max_{x\in\{0,1\}^N}\{u^0(x) - p\cdot x + \textstyle\sum_{i\in N} p_i\} \\ &= \arg\max_{x\in\{0,1\}^N}\{u^0(x) - p\cdot x\}. \end{aligned}$$

The set $D^0(p)$ contains those bundles that the seller wishes to keep in hand and give her the highest revenues. Although the seller has a different objective from the bidders, her revenue-maximizing behavior is similar to a bidder's profit-maximizing behavior. Observe that if $x \in D^0(p)$ at prices p, the seller will retain the bundle x and sell all other items by receiving the payment of $p \cdot (\sum_{i\in N} e(i) - x) = \sum_{i\in N} p_i - p \cdot x$.

Definition 1 (Competitive or Walrasian Equilibrium) A *competitive or Walrasian equilibrium* (p, X) consists of a price vector $p \in \mathbb{R}^N_+$ and an allocation X such that $x^j \in D^j(p)$ for every $j \in B_0$.

If (p, X) is a competitive equilibrium, we call p an *equilibrium price vector* and X an *equilibrium allocation.* We say that X is supported by p. It is well-known from the first welfare theorem that every equilibrium allocation is efficient.

Baldwin and Klemperer (2019) have recently proposed a powerful necessary and sufficient condition for the existence of competitive equilibrium in an exchange economy with indivisible commodities, which will be used in our auction. Their condition (i.e. (A2) below) covers and generalizes many previous conditions including the widely-used gross substitutes condition of Kelso and Crawford (1982).

bundle is below her reserve price, and she will keep any bundle of her own items if the bundle is not sold.

Let $\sharp A$ denote the cardinality of any given finite set A. The dimension of any given set $A \subset \mathbb{R}^N$ is understood as the dimension of the affine span of A. With respect to any given utility function $u : S \to \mathbb{R} \cup \{-\infty\}$ with a finite set $S \subset \mathbb{Z}^N$ and $\sharp \mathrm{dom}(u) > 1$, let the demand set at a price vector $p \in \mathbb{R}^N$ be given by

$$D_u(p) = \arg\max_{x \in S}\{u(x) - p \cdot x\}.$$

Note that the domain S of the function u is very general, not restricted to the set $\{0,1\}^N$. Following Baldwin and Klemperer (2019), we introduce the locus of indifference prices, demand type and unimodular demand type. We say that the set

$$\mathcal{T}_u = \{p \in \mathbb{R}^N \mid \sharp D_u(p) > 1\}$$

is *the locus of indifference prices* (LIP) of the demand mapping D_u. This set $\mathcal{T}_u$ concerns those price vectors p at which there are at least two optimal bundles for any agent who has the utility function u. LIP contains the only prices at which demand can change in response to a price change, and is the union of $(n-1)$-dimensional polyhedral pieces called facets(a facet of a polyhedron of dimension n is a face that has dimension $n-1$). These facets separate the unique demand regions, in each of which some bundle is the unique demand; see Baldwin and Klemperer (2019, Prop. 2.4). The *normal vector* to a facet F is a vector which is perpendicular to F at a point in its relative interior. A non-zero integer vector is *primitive* if the greatest common divisor of its coordinates is one.

Definition 2 (Demand Type) A finite set $\mathcal{D}$ of primitive vectors in $\mathbb{Z}^N$ is a *demand type* of function u if $v \in \mathcal{D}$ implies $-v \in \mathcal{D}$ and every facet of the LIP $\mathcal{T}_u$ has its normal vector in $\mathcal{D}$.

By definition, a demand type may contain vectors which are not a normal vector of any facet of LIP $\mathcal{T}_u$.

A square matrix is *unimodular* if all its elements are integral and its determinant is $+1$ or -1. A matrix M is *totally unimodular* if every minor of M is 0 or ± 1. A set of n integer vectors in $\mathbb{R}^N$ is *a unimodular basis* for $\mathbb{R}^N$ if the $n \times n$ matrix which has the n integer vectors as its columns is unimodular.

Definition 3 (Unimodular Demand Type) A demand type $\mathcal{D}$ is *unimodular* if every linearly independent subset of $\mathcal{D}$ can be extended to a unimodular basis for $\mathbb{R}^N$.

For a unimodular demand type $\mathcal{D}$, additional vectors required to form a unimodular basis are possibly chosen from outside $\mathcal{D}$. Note that unimodular demand types are derived from utility functions and given as sets of integer vectors associated with unimodular matrices. These demand types capture the essential and natural attributes of the commodities but do not reveal the values of the consumers. For instance, consumers view tables as something sharing the same physical property but they can each have different valuations on tables.

Proposition 1 *Every unimodular demand type can be added with less than n new vectors so that the enlarged set is still a unimodular demand type and contains at least one basis.*

This new and basic property of unimodular demand types is used in our auction design by naturally assuming that every given unimodular demand type spans the space $\mathbb{R}^N$. As the concept of demand type is quite new, we give an example to illustrate it.

Example 1 There is a market where the seller wishes to sell two items a and b to three bidders. Every agent knows her values privately. We consider two possibilities. Case 1: Both items are substitutes. Agents' valuations are given in Table 1. Case 2: Both items are complements. Agents' valuations are given in Table 2.

Table 1: The case of substitutes.

Agents\Bundles	$\emptyset$	a	b	ab
Bidder 1	0	3	4	5
Bidder 2	0	5	2	6
Bidder 3	0	3	3	4
Seller	0	2	2	3

Table 2: The case of complements.

Agents\Bundles	$\emptyset$	a	b	ab
Bidder 1	0	2	2	5
Bidder 2	0	2	2	5
Bidder 3	0	1	1	4
Seller	0	1	1	3

For this example, in the case of substitutes, all agents have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$. The locus of indifference prices of bidder 1 is shown in Figure 1. In the case of complements, all agents have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$. The locus of indifference prices of bidder 1 is shown in Figure 2. In the two figures, $(1,0)$ stands for item a, $(0,1)$ for item b and $(1,1)$ for two items ab, and the normal vector of every facet of the LIP $\mathcal{T}_{u^1}$ is the dashed line.

The following two assumptions are imposed on our auction model $\mathcal{M}$:

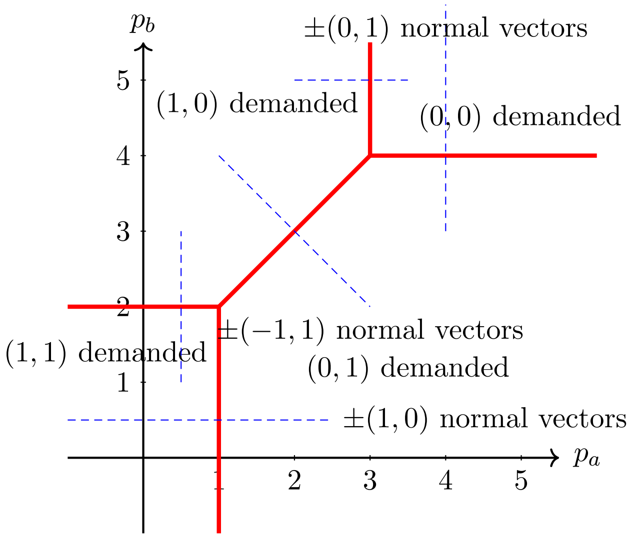

Figure 1: $u^1(0,0) = 0$, $u^1(1,0) = 3$, $u^1(0,1) = 4$, and $u^1(1,1) = 5$. The five connected lines denote LIP.

(A1) *Integer Private Values*: Every agent $j \in B_0$ knows her own utility function $u^j : \{0,1\}^N \to \mathbb{Z} \cup \{-\infty\}$ privately.

(A2) *Common Unimodular Demand Type*: All agents $j \in B_0$ have the same unimodular demand type $\mathcal{D}$ for their utility functions u^j.

Assumption (A1) means that every agent treats her valuation as her private, personal information. The integer-valued assumption is a standard and natural one, as people valuate the bundles of goods in units of currency, say, in dollars, which cannot be closer to the nearest penny. As every agent's utility function is assumed to be private information, this means that the agent possessing this information can make use of it in a way it is in her best interest. However, it is typically assumed that the seller acts truthfully while bidders may behave strategically (see e.g., Ausubel 2004, 2006 and Perry and Reny 2005), because it is well-known from Myerson and Satterthwaite (1983) that even in a simple bilateral trading model with one seller, one buyer and one item, it is impossible to achieve efficiency, individual rationality and strategy-proofness for both the seller and the buyer; see also Krishna (2002). Unlike many previous models, we allow the seller to have her personal reserve price function u^0. This makes the model more realistic and practical.

Assumption (A2) can be alternatively stated as the union of the demand types of all agents $j \in B_0$ is a unimodular demand type. This assumption says that agents may have quite different valuations on every bundle of items but they all have the same demand

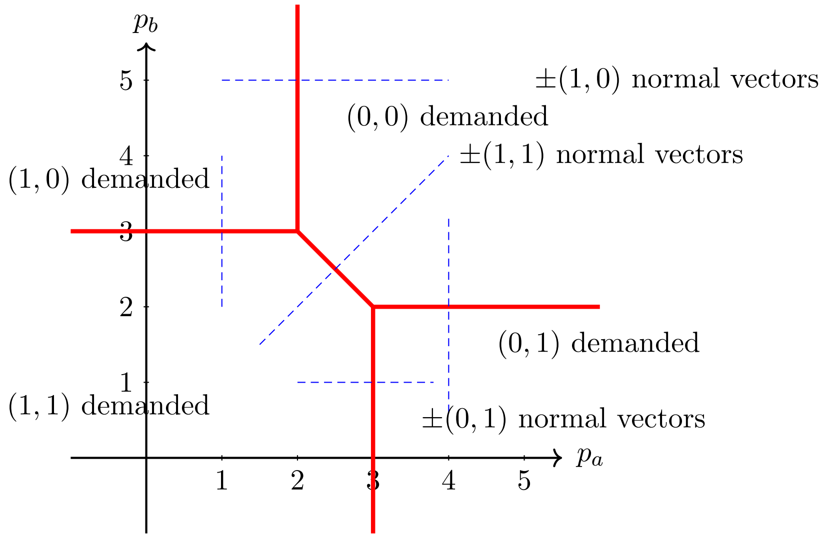

Figure 2: $u^1(0,0) = 0$, $u^1(1,0) = u^1(0,1) = 2$, and $u^1(1,1) = 5$. The five connected lines denote LIP.

type, which captures the quintessence of the items.[4] It is a test condition imposed upon every individual agent, which is neat and easy to check compared with the earlier necessary and sufficient conditions introduced by Bikhchandani and Mamer (1997), Ma (1998), Sun and Yang (2002), and Yang (2003) which are given as aggregated conditions on the entire market. Unimodular demand types are numerous. Note that for the market described in Footnote 3, Assumption (A2) and $\text{dom}(u^0) = \{0,1\}^N$ imply that the unimodular demand type $\mathcal{D}$ shared by all agents is totally unimodular as it contains all unit vectors $e(i)$, $i \in N$.

We now briefly discuss three typical and important classes of unimodular demand types given by Baldwin and Klemperer (2019). Note that besides these three classes there are numerous other classes of demand types which remain to be explored.

Definition 4 (Gross Substitutes) A demand type $\mathcal{D}$ is *gross substitutes* (GS) or simply *substitutes* if every vector $x \in \mathcal{D}$ has at most one 1 entry and at most one -1 entry and no other nonzero entries.

This definition captures the gross substitutes or simply substitutes condition of Kelso and

[4]Baldwin and Klemperer (2019, Theorem 4.3) have shown in a nonconstructive way by tropical geometry and convex analysis that Assumption (A2) is a necessary and sufficient condition for the existence of competitive equilibrium without requiring integral valuations whose equilibrium prices can be any real numbers so may not be integral. Tran and Yu (2019) proposed an alternative proof of Theorem 4.3 of Baldwin and Klemperer through the linear programming approach and a sealed-bid product-mix auction. Baldwin et al. (2020) examined a general model with income effects.

Crawford (1982) on the demand behavior. See Gul and Stacchetti (1999, 2000), Fujishige and Yang (2003), Hatfield and Milgrom (2005), Ausubel (2006), Milgrom and Strulovici (2009), Shioura and Tamura (2015), Murota et al. (2016) for various results on substitutes.

Definition 5 (Gross Substitutes and Complements) Assume that S_1 and S_2 are disjoint subsets of N and their union equals N. A demand type $\mathcal{D}$ is *gross substitutes and complements* (GSC) if every vector $x \in \mathcal{D}$ has at most two nonzero entries of $+1$ or -1 and no other nonzero entries so that if two nonzero entries of x have the same sign, then one nonzero component must be indexed by an element in S_1 and the other must be indexed by an element in S_2.

GSC says that items in either S_1 or S_2 are substitutes but items across the two sets are complementary. Observe when either S_1 or S_2 becomes empty, GSC coincides with GS and thus generalizes GS. This condition corresponds to the one in Sun and Yang (2006, 2009) as a generalization of gross substitutes. See also Shioura and Yang (2015).

Definition 6 (Unimodular Complements) A demand type $\mathcal{D}$ is *unimodular complements* if $x \in \mathcal{D}$ implies either $x \in \{0, 1\}^N$ or $x \in \{0, -1\}^N$ and $\mathcal{D}$ is unimodular.

A basis change is called *a unimodular transformation* if we have $y = Ax$ for every $x \in \mathbb{R}^N$ and A is a unimodular matrix of order n. Baldwin and Klemperer (2019, Prop. 6.2; 2014, Theorem 5.27) have shown that every unimodular demand type is a unimodular transformation of some unimodular complements demand type. This means that unimodular complements demand types are so rich that any other unimodular demand type can be obtained from them. It is known that the gross substitutes condition is the most general representation of substitutability for equilibrium; see Gul and Stacchetti (1999, Theorem 2, p. 103). However, we cannot have a similar statement for complements, because unimodular complements demand types are numerous and varied and there is no unique maximal unimodular complements demand type.

In the literature, we have a far better understanding of substitutes than of complements. Levin (1997) introduced an optimal sealed-bid auction for two complementary items, extending the auction of Myerson (1981) for a single item. Sun and Yang (2014) proposed a dynamic auction for multiple complements that satisfy super-additivity. Their model does not guarantee the existence of competitive equilibrium (with linear pricing) so anonymous and nonlinear pricing has to be used. In this case the complements demand type is not unimodular.

3 On the Structure of Competitive Equilibria

In this section we present several basic results which will play an important role in our auction design and analysis.[5] These results are also interesting on their own right, intuitive, and economically meaningful.

We first introduce several mathematical concepts. Other concepts can be found in the appendix. A set $S \subseteq \mathbb{R}^N$ is a *polyhedron* if $S = \{x \in \mathbb{R}^N \mid Ax \leq b\}$ for some $m \times n$ matrix A and an m-vector b. A bounded polyhedron is called a *polytope*. A polyhedron $S \subseteq \mathbb{R}^N$ having at least one vertex is *integral* if all its vertices are integral. The *Minkowski sum* of any two sets S and T in $\mathbb{R}^N$ is defined as $S + T = \{x + y \mid x \in S,\ y \in T\}$. Given any $x, y \in \mathbb{R}^N$, define their meet $x \wedge y$ as the componentwise minimum of x and y and join $x \vee y$ as the componentwise maximum of x and y. A set $S \subset \mathbb{R}^N$ is *a lattice* if $x \wedge y \in S$ and $x \vee y \in S$ for any $x, y \in S$. A polyhedron is called *a polyhedron with a lattice structure* if it is also a lattice. It is known that a lattice is not necessarily a polyhedron. A function f defined on a convex set S in $\mathbb{R}^N$ is called *a polyhedral convex function* if it is given as

$$f(x) = \max\{B_j \cdot x + c_j \mid j = 1, \cdots, k\} \qquad (x \in S),$$

where B_j is an n-vector and c_j is a constant, $j = 1, \cdots, k$ for a given positive integer k.

Let us turn to our auction model. For every agent $j \in B_0$, define her indirect utility function $V^j : \mathbb{R}^N \to \mathbb{R}$ by

$$V^j(p) = \max_{x \in \{0,1\}^N} \{u^j(x) - p \cdot x\} \tag{2}$$

and, for the market model, define the Lyapunov function $\mathcal{L} : \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{L}(p) = \sum_{i \in N} p_i + \sum_{j \in B_0} V^j(p) \tag{3}$$

where V^j is the indirect utility function of agent $j \in B_0$. This type of function is well-known in the literature for economies with divisible goods (see e.g., Arrow and Hahn 1971

[5]Precisely, they are crucial to the analysis of the convergence and other properties of our dynamic auctions in Sections 4 and 5. The description of the auctions, however, dose not depend on this section.

and Varian 1981) and has been explored by Ausubel (2006) and Sun and Yang (2009) for auction markets with indivisible goods. We have the following two basic results.

Lemma 1 *For any given function* $f : S \to \mathbb{R}$ *with a nonempty finite set* $S \subset \mathbb{Z}^N$*, the function* $g : \mathbb{R}^N \to \mathbb{R}$ *defined by* $g(p) = \max_{x \in S}\{f(x) - p \cdot x\}$ *for every* $p \in \mathbb{R}^N$ *is a decreasing polyhedral convex function.*

Lemma 2 *For the market model, the Lyapunov function* $\mathcal{L}$ *defined by* (3) *is a polyhedral convex function bounded from below.*

The above two lemmas are very general and do not depend on any particular assumptions such as Assumptions (A1) and (A2). Proposition 1 of Ausubel (2006) and Lemma 1 of Sun and Yang (2009) imply that $p \in \mathbb{R}^N$ is an equilibrium price vector if and only if it is a minimizer of the Lyapunov function $\mathcal{L}$ provided that the market has an equilibrium.

To study the collective behavior of all agents $j \in B_0$, we consider the *convolution* u of their utility functions u^j given by

$$u(x) = \max\{\sum_{j \in B_0} u^j(y^j) \mid x = \sum_{j \in B_0} y^j \text{ where } y^j \in \{0,1\}^N \text{for every } j \in B_0\} \tag{4}$$

for every $x \in \{0, 1, \cdots, m+1\}^N$. This function is closely related to the Lyapunov function $\mathcal{L}$. For all $p \in \mathbb{R}^N$ and all $x^j \in D^j(p)$ of agents $j \in B_0$, define

$$g(x) = \sum_{j \in B_0} u^j(x^j) \text{ where } x = \sum_{j \in B_0} x^j. \tag{5}$$

From all demand sets $D^j(p)$ we obtain the following demand set of Minkowski sum

$$D^{Ms}(p) = D^0(p) + D^1(p) + \cdots + D^m(p). \tag{6}$$

When an agent's demand type $\mathcal{D}$ with respect to utility function u is unimodular, we say that the agent has a *UDT* $\mathcal{D}$ *utility function* u. The following two results demonstrate several basic properties of the function g of (5) and the Minkowski sum D^{Ms} of (6), playing an important role in our auction analysis. Related to our Lemma 4 is Corollary 3.14 of Baldwin and Klemperer (2019, p. 886).

Lemma 3 *For any integer-valued UDT* $\mathcal{D}$ *utility function* $u : S \to \mathbb{Z} \cup \{-\infty\}$ *with a finite set* $S \subset \mathbb{Z}^N$ *and* $\sharp\text{dom}(u) > 1$*, if the convex hull of the demand set* $D_u(p)$ *for a price vector* p *is full-dimensional, the price vector* p *must be integral and unique.*

Lemma 4 *Under Assumptions* (A1) *and* (A2), *the function* g *of* (5) *is well-defined, coinciding with the convolution function* u *of* (4) *and being discrete concave with the unimodular demand type* $\mathcal{D}$. *In particular, the Minkowski sum* $D^{Ms}(p)$ *of* (6) *is the demand set for valuation* $g = u$ *and has the same unimodular demand type* $\mathcal{D}$.

We are ready to establish our first major result on the set of competitive equilibrium price vectors, which exhibits an elegant geometric structure, extending and sharpening the classic lattice results of Shapley and Shubik (1971) on assignment models, Gul and Stacchetti (1999) and Ausubel (2006) on gross substitutes models. This theorem ensures that our proposed auction will terminate with an integer equilibrium price vector no matter which integer vector it starts with (see Theorem 3 in Section 4).

Theorem 1 *Under Assumptions* (A1) *and* (A2), *the set of competitive equilibrium price vectors forms a nonempty integral polytope.*

For gross substitutes, we can show that the set of competitive equilibrium price vectors forms a nonempty integral polytope with a lattice structure.

4 Basic Dynamic Auction Design

In this section we consider the basic case that bidders bid straightforwardly as price-takers. We propose a universally convergent dynamic (UCD) auction that applies to *all unimodular demand types.* This section prepares us to deal with a more natural and more realistic situation in Section 5 where bidders have market power, may therefore act strategically rather than sincerely as price-takers, and may also occasionally make mistakes. Based on the UCD auction we will introduce in Section 5 an efficient and strategy-proof dynamic auction that allows bidders to learn, adjust, and correct.

In a dynamic auction, at each time $t \in \mathbb{Z}_+$, the auctioneer announces a price for every item and then every bidder chooses a bid. We introduce the concept of sincere bidding.

Definition 7 (Sincere Bidding) Agent $j \in B_0$ *bids sincerely* with respect to her utility function u^j if she always submits a bid $B^j(t)$ equal to her demand set $D^j(p(t)) = \arg\max_{x \in \{0,1\}^N} \{u^j(x) - p(t) \cdot x\}$ at every time $t \in \mathbb{Z}_+$ and any price vector $p(t) \in \mathbb{R}^N$.

Roughly speaking, our universally convergent dynamic auction works as follows: At each time $t \in \mathbb{Z}_+$, the auctioneer announces the current prices $p(t) \in \mathbb{Z}^n$ and every bidder

j responds by reporting his demand $D^j(p(t))$. Then she uses every bidder's reported demand $D^j(p(t))$ to search for a price adjustment δ in a neighborhood of prices $p(t)$ in order to update the current prices. To do so, she tries to reduce the value of the Lyapunov function $\mathcal{L}(p(t)+\delta)$ as much as possible, until a minimizer of the Lyapunov function, i.e., a competitive equilibrium price vector, is found.

We now introduce the concept of a search set which is a key building block of our auction design and gives an appropriate neighborhood of the current prices for price adjustment. The search set is defined with respect to any demand type $\mathcal{D}$ as given in Assumption (A2).

Definition 8 (Search Set) For any given demand type $\mathcal{D}$, its *search set* denoted by $\mathcal{SD}$ is the collection of the zero vector and all nonzero primitive integer vectors $\delta \in \mathbb{Z}^N$ such that we have $\delta \cdot d_j = 0$ for some $n-1$ linearly independent vectors $d_1, \cdots, d_{n-1} \in \mathcal{D}$.

One may view the search set as a family of the zero vector and all nonzero primitive integer vectors $\delta \in \mathbb{Z}^N$ such that δ is a normal vector of a facet of a full-dimensional convex hull of a demand set at some price vector p. The search set is a spanning set of $\mathbb{R}^N$, can be easily obtained from any given demand type and varies from one demand type to another. It applies universally to all kinds of commodities regardless of whether they are substitutes or complements or anything else. The search set $\mathcal{SD}$ will be used as a litmus test of optimality and for local searches. More precisely, it will be shown that $p(t) \in \mathbb{Z}^N$ is a minimizer of the Lyapunov function $\mathcal{L}$ if and only if $\mathcal{L}(p(t)) \leq \mathcal{L}(p(t)+\delta)$ for all $\delta \in \mathcal{SD}$, and that if $p(t) \in \mathbb{Z}^N$ is not a minimizer of $\mathcal{L}$, we must have $\mathcal{L}(p(t)) > \mathcal{L}(p(t)+\delta)$ for some $\delta \in \mathcal{SD}$. These properties play a pivotal role in our basic dynamic auction design.

It will be helpful to use the simple case of complements in Example 1 to illustrate why a typical multi-item ascending/English auction can be plagued by the exposure problem and how our new auction overcomes the problem and succeeds in finding a competitive equilibrium. Let us first see how a multi-item English auction would operate. The seller initially announces low prices $p(0) = (p_a(0), p_b(0)) = (0, 0)$. Clearly, every agent demands the two items. As the bundle ab is overdemanded, the auction will raise the two prices simultaneously, say each by one unit, an integer increment as a typical English auction does. The price vector is updated to $p(1) = (1, 1)$. At $p(1)$, ab is still overdemanded and the prices are raised up to $p(2) = (2, 2)$. At $p(2)$, ab is still overdemanded and the price is updated to $p(3) = (3, 3)$. At $p(3)$ no bidder wants to demand any item and the auction has stuck in a non-equilibrium state. This phenomenon is called the exposure problem; see e.g., Milgrom (2000).

Now we will see how our basic auction resolves the exposure problem. As it will be shown below, at any time $t \in \mathbb{Z}_+$, in order to reduce the value of the Lyapunov function, the auctioneer/seller just needs to adjust the current prices $p(t)$ to the next prices $p(t+1) = p(t) + \delta(t)$ by finding an optimal search direction $\delta(t)$ to the following problem until the vector of zeros becomes an optimal solution:

$$\max_{\delta \in \mathcal{SD}} \{ \sum_{j \in B_0} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{i \in N} \delta_i \} \tag{7}$$

This example has the demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$ and its search set $\mathcal{SD} = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,-1)\}$. Starting with $p(0) = (p_a(0), p_b(0)) = (0,0)$, the auctioneer updates prices $p(t+1) = p(t) + \delta(t)$ according to (7). At $p(0)$, the bundle ab is demanded by every agent. In this case, there are two optimal adjustments $(1, 0)$ and $(0, 1)$ and we can choose either of the two. The auction process is shown in Table 3. The auction stops at $p(5) = (3, 2)$ and finds a Walrasian equilibrium in which ab is allocated to bidder 1 or 2 who pays 5 in return, and other bidders get nothing and pay nothing. Note that the auction can also stop at $(2, 3)$ if one chooses $\delta(4) = (0, 1)$ at $p(4) = (2, 2)$.

Table 3: Illustration of the New Basic Auction.

time t	prices $p(t)$	$\delta(t)$	$D^0(p(t))$	$D^1(p(t)) = D^2(p(t))$	$D^3(p(t))$
0	$(0,0)$	$(1,0)$	$\{ab\}$	$\{ab\}$	$\{ab\}$
1	$(1,0)$	$(0,1)$	$\{ab\}$	$\{ab\}$	$\{ab\}$
2	$(1,1)$	$(1,0)$	$\{ab\}$	$\{ab\}$	$\{ab\}$
3	$(2,1)$	$(0,1)$	$\{ab, b, \emptyset\}$	$\{ab\}$	$\{ab\}$
4	$(2,2)$	$(1,0)$	$\{\emptyset\}$	$\{ab\}$	$\{ab, \emptyset\}$
5	$(3,2)$	$(0,0)$	$\{\emptyset\}$	$\{ab, b, \emptyset\}$	$\{\emptyset\}$

The underlying principle of our auction is to find a minimizer of the nonlinear Lyapunov function $\mathcal{L}$, although we will not be able to use the function $\mathcal{L}$ directly, because the utility function of every bidder is private information and unavailable to the auctioneer.

Moreover, because the function $\mathcal{L}$ is nonlinear, we cannot reach any of its minimizers in one step but have to do it iteratively by local search. Before discussing our dynamic auction in detail, we first give the blueprint for our auction design:

- First, at current prices $p(t) \in \mathbb{Z}^N$ for time $t \in \mathbb{Z}_+$, the auctioneer searches locally for a price adjustment $\delta(t)$ in the convex hull $\text{Conv}(\mathcal{SD})$ of the search set $\mathcal{SD}$ to reduce the value of the Lyapunov function $\mathcal{L}$ as much as possible from $\mathcal{L}(p(t))$ to $\mathcal{L}(p(t)+\delta(t))$. We will show that this local search can be done over the much easier finite set $\mathcal{SD}$ instead of over the complicated, dense and infinite set $\text{Conv}(\mathcal{SD})$.

- Second, we will show that based on every bidder j's reported demand set $D^j(p(t))$ at prices $p(t)$, the auctioneer's solving the unobservable maximization problem of $\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta(t))$ over the search set $\mathcal{SD}$ amounts to solving the much easier observable problem (7). This process will be repeated until a minimizer of the Lyapunov function, i.e., a competitive equilibrium price vector, is found.

Our auction can be seen as a substantial generalization of those of Demange et al. (1986), Gul and Stacchetti (2000), and Ausubel (2006) from gross substitutes to all unimodular demand types and is particularly close to Ausubel's auction (2006, pp. 618-619). However, we cannot generalize or use their arguments directly but have to explore quite different and general techniques that apply to all unimodular demand types. The following lemma will be used to show a crucial result, Proposition 2 given below.

Lemma 5 *Let $\mathcal{SD}$ be the search set of a unimodular demand type $\mathcal{D}$ and $\delta \in \mathcal{SD}$ be a primitive normal vector of an $(n-1)$-dimensional space spanned by $d_1, \cdots, d_{n-1} \in \mathcal{D}$. If $d_1, \cdots, d_{n-1}, d_n \in \mathcal{D}$ form a basis, we have $\alpha|\delta \cdot d_n| = 1$ for some $\alpha \geq 1$.*

The next proposition concerning the Lyapunov function shows that the nonlinear optimization problem (8) over the convex hull of the finite search set is equivalent to the nonlinear optimization problem (8) over the finite search set. This implies that when the auctioneer tries to adjust prices, she just needs to focus on the few choices in the search set $\mathcal{SD}$ rather than gropes around the entire convex hull of the search set $\mathcal{SD}$.

Proposition 2 *Under Assumptions* (A1) *and* (A2), *for any $p(t) \in \mathbb{Z}^N$ we have*

$$\max_{\delta \in \text{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} = \max_{\delta \in \mathcal{SD}} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\}. \tag{8}$$

From the above result and its proof (given in the Appendix), we immediately have two corollaries. The first says that the optimal solutions of the constrained nonlinear optimization problem (8) exist and correspond to the vertices of the set of all optimal solutions. The second roughly says that if we can change prices slightly, the demand set of every bidder will not change.

Corollary 1 *Under Assumptions* (A1) *and* (A2)*, the set of solutions to the left-side problem of* (8) *is a nonempty integral polytope.*

Corollary 2 *Under Assumptions* (A1) *and* (A2)*, then for any* $j \in B_0$*, any* $p \in \mathbb{Z}^N$*, and any* $\delta \in \mathcal{SD}$*, we have* $D^j(p + \varepsilon\delta) \subseteq D^j(p)$ *for all* $\varepsilon \in [0, 1)$ *and* $x^j \in \arg\min_{x \in D^j(p)} x \cdot \delta$ *lies in* $D^j(p + \varepsilon\delta)$ *for all* $\varepsilon \in [0, 1]$*.*

Corollary 2 substantially generalizes Proposition 2 of Ausubel (2006) on gross substitutes to all unimodular demand types.

The following result gives a powerful local characterization of optimality or competitive equilibrium price vectors, saying that the search set $\mathcal{SD}$ is a simple test set for verifying whether a point is a minimizer of the Lyapunov function $\mathcal{L}$ or not. Recall that because the set of competitive equilibrium price vectors in our auction market is a nonempty integral polytope by Theorem 1, an n-vector p^* is a competitive equilibrium price vector if and only if it is a minimizer of the Lyapunov function $\mathcal{L}$.

Theorem 2 *Under Assumptions* (A1) *and* (A2)*,* $p^* \in \mathbb{Z}^N$ *is a minimizer of the Lyapunov function* $\mathcal{L}$ *in* (3) *if and only if* $\mathcal{L}(p^*) \leq \mathcal{L}(p^* + \delta)$ *for all* $\delta \in \mathcal{SD}$*.*

Finding an optimal solution of a nonlinear problem usually cannot be done in one step but requires multiple successive local searches. Our next corollary says that if the minimum of the nonlinear Lyapunov function $\mathcal{L}$ has not been reached, one can further reduce the function value along directions in the search set. Clearly, one can repeat such local searches.

Corollary 3 *Under Assumptions* (A1) *and* (A2)*, if* $p \in \mathbb{Z}^N$ *is not a minimizer of the Lyapunov function* $\mathcal{L}$ *in* (3)*, it holds* $\mathcal{L}(p + \delta) < \mathcal{L}(p)$ *for some* $\delta \in \mathcal{SD}$*.*

Now we can discuss the universally convergent dynamic auction in detail. Starting with an arbitrarily given price vector $p(t) \in \mathbb{Z}^N$, the auction tries to solve the following maximization problem with the unobservable Lyapunov function $\mathcal{L}$

$$\max_{\delta \in \text{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\}. \tag{9}$$

It follows from Proposition 2 that the continuous maximization problem over the entire convex hull of the search set $\mathcal{SD}$ can be considerably reduced to the following discrete optimization problem over the finite set $\mathcal{SD}$ of integer vectors:

$$\max_{\delta \in \mathcal{SD}} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\}. \tag{10}$$

The maximand of (10) can be further written as

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta) = \sum_{j \in B_0} (V^j(p(t)) - V^j(p(t)+\delta)) - \sum_{i \in N} \delta_i. \tag{11}$$

Observe that the above formula involves every bidder's valuation of every bundle of items, so it involves private information. Apparently, it is impossible for the auctioneer to know such information unless the bidders are willing to tell her. Fortunately, by Corollary 2 above she can immediately infer the difference between $\mathcal{L}(p(t))$ and $\mathcal{L}(p(t)+\delta)$ just from the reported demands $D^j(p(t))$ and the price variation δ because $D^j(p(t)) \supseteq D^j(p(t)+\varepsilon\delta)$ for all $j \in B$ and all $\varepsilon \in [0,1)$. In fact, when prices move from $p(t)$ to $p(t)+\delta$, the reduction in indirect utility for bidder j is uniquely given by

$$V^j(p(t)) - V^j(p(t)+\delta) = \min_{x^j \in D^j(p(t))} x^j \cdot \delta. \tag{12}$$

Consequently, equation (11) becomes the following simple formula whose right side involves only price variation δ and optimal choices at $p(t)$:

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta) = \sum_{j \in B_0} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{i \in N} \delta_i. \tag{13}$$

From the above discussion, Proposition 2 and Corollary 2, we obtain the next crucial proposition regarding problem (9).

Proposition 3 *Under Assumptions* (A1) *and* (A2), *for any* $p(t) \in \mathbb{Z}^N$ *we have*

$$\max_{\delta \in \text{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} = \max_{\delta \in \mathcal{SD}} \{\sum_{j \in B_0} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{i \in N} \delta_i\}. \tag{14}$$

Note that the above formula shows a dramatic change from the unobservable Lyapunov function $\mathcal{L}$ to the observable reported demands of bidders and integer price adjustment δ. The right-hand max-min formula admits an intuitive and interesting interpretation:

- When the auctioneer adjusts the prices from $p(t)$ to $p(t+1) = p(t) + \delta(t)$, she tries to balance two opposing forces by minimizing every bidder's loss for every possible price change δ in the search set $\mathcal{SD}$ and choosing one price change from all possible price changes that maximizes the seller's gain.
- In the auction process bidders do nothing but report their demand sets $D^j(p(t))$ and the auctioneer adjusts prices according to the right-hand formula of (14).

Formally, we can give the detailed steps of the auction as follows:

The Universally Convergent Dynamic (UCD) Auction

Step 1: The auctioneer announces an (arbitrary) initial integer price vector $p(0) \in \mathbb{Z}^N$. Let $t := 0$ and go to **Step 2**.

Step 2: Every agent $j \in B_0$ reports her demand $D^j(p(t))$ at $p(t)$ to the auctioneer. Based on reported demands $D^j(p(t))$, the auctioneer calculates an optimal solution $\delta(t)$ to the righthand problem of (14). As soon as the vector $\mathbf{0}$ of zeros is an optimal solution to the problem, the auction stops. Otherwise, the auctioneer updates $p(t+1) := p(t) + \delta(t)$ and $t := t + 1$. Return to **Step 2**.

Note that this auction may run in several forms including ascending, descending, or both. In principle, in which form the auction operates hinges upon the search set of the underlying demand type, the starting prices, and the time. We now have

Theorem 3 *Under Assumptions* (A1) *and* (A2)*, starting with any given initial integer price vector* $p(0) \in \mathbb{Z}^N$*, the UCD auction finds an integer competitive equilibrium vector in a finite number of rounds.*

Observe that the above theorem is very general and holds for all unimodular demand types. This means that items can be substitutes, complements, or possess any other possible properties beyond substitutability or complementarity. The proof of the theorem makes use of mainly convexity and unimodularity and does not invoke the familiar submodularity. In the literature, submodularity is commonly used for the convergence of auction; see Gul and Stacchetti (2000) and Ausubel (2006). It is known from Ausubel and Milgrom (2002) that items are (gross) substitutes to a bidder if and only if the bidder's indirect utility function is submodular. Therefore, for gross substitutes, the Lyapunov function must be submodular. Substitutes are closely related to submodularity and complements are related to supermodularity. Besides gross substitutes, there are so many other different demand types which may not have a clear-cut property like substitutes or complements and thus the corresponding Lyapunov function can be neither submodular nor supermodular. As a result, it is natural and logical that the proof of the above theorem relies mainly on convexity and unimodularity and cannot and do not use submodularity.

We now discuss the familiar ascending or descending auctions for the gross substitutes of which we have had a far better understanding than of any other type; see e.g., Kelso and Crawford (1982), Demange et al. (1986), Gul and Stacchetti (2000), Milgrom (2000), and Crawford (2008) whose auctions are all ascending, and Ausubel (2006) whose auction can be ascending or descending. Note that the well-known assignment or unit-demand market models (see e.g., Crawford and Knoer 1981 and Demange et al. 1986) are special instances of gross substitutes. Let $\mathcal{D}$ be the gross substitutes demand type given in Definition 4 of Section 2. Then we have its search set $\mathcal{SD} = \{0,1\}^N \cup \{0,-1\}^N$ which has a clear-cut structure. Let $\Delta = \{0,1\}^N$ and let $\bar{\Delta}$ be the convex hull of the set Δ. Let $\Delta^* = -\Delta$ and $\bar{\Delta}^* = -\bar{\Delta}$. If we use the search set Δ in the UCD auction and set the initial prices $p(0)$ so low that all the items are demanded by every agent, the auction is an ascending one and can find the minimum Walrasian equilibrium prices. If we use the search set Δ^* and set the initial prices $p(0)$ so high that none of the items is demanded by any agent, the auction is a descending one and can find the maximum Walrasian equilibrium prices. In the case of gross substitutes, our UCD auction is similar to Ausubel's and Gul and Stacchetti's in the ascending format and similar to Ausubel's in the descending format.

Note that Klemperer (2008, 2010, 2018) proposed sealed-bid product-mix auctions for substitutes.

5 Dynamic Auction Design with Strategic Bidders

In Section 4 we assume that every agent acts as a price-taker. In this section we drop that assumption by considering a more natural and more realistic environment where bidders are strategic and may therefore act strategically, and they may also occasionally make mistakes. We investigate how we should expect such individuals to behave and how to prevent their possible manipulation and miscalculation and how to allow them to learn, adjust, and correct if they make mistakes or behave badly. To address these questions, we develop an efficient, incentive-compatible dynamic auction mechanism built upon the basic dynamic auction introduced in the previous section.

Recall that $\mathcal{M}$ stands for the (original) market with m bidders and the seller with the set N of n items. For every bidder $j \in B$, let $\mathcal{M}_{-j}$ denote the market $\mathcal{M}$ *without the participation of bidder* j and $B_{-j} = B_0 \setminus \{j\}$. For convenience, we set $\mathcal{M}_{-0} = \mathcal{M}$ and $B_{-0} = B_0$. So, for every $k \in B_0$, market $\mathcal{M}_{-k}$ comprises the set B_{-k} of agents and the set N of n items. The seller always participates in every market and is not strategic.

The following defines the Vickrey-Clarke-Groves (VCG) mechanism; see Vickrey (1961), Clarke (1971), and Groves (1973). The definition given below is more general than its standard one because we permit the seller to have a reserve function; see Ausubel and Cramton (2004) on a similar extension for divisible goods. The standard one assumes that the seller values everything at zero. Recall that $R(N)$ denotes the market value of the items in N for the market $\mathcal{M}$. Let $R_{-j}(N)$ represent the market value of the items in N in the market $\mathcal{M}_{-j}$ for every $j \in B$ based on the reported u^j $(j \in B_0)$.

Definition 9 (VCG Mechanism) The *VCG mechanism* is the following procedure: Every agent $j \in B_0$ reports her value function u^j. The auctioneer computes an efficient allocation X with respect to all reported u^j and assigns bundle x^j to bidder $j \in B$ and charges him a payment of $\beta^*_j = u^j(x^j) - R(N) + R_{-j}(N)$, where $R(N)$ and $R_{-j}(N)$ are the market values of the items in N in the markets $\mathcal{M}$ and $\mathcal{M}_{-j}$ for all $j \in B$, respectively. Bidder j's VCG payoff equals $R(N) - R_{-j}(N)$, $j \in B$.

It is known from Green and Laffont (1977) and Holmström (1979) that in the setting of transferable utility any strategy-proof mechanism must generate the VCG outcome. As discussed earlier, strategy-proof dynamic auctions have distinct advantages over those of sealed-bid. In the case of a single item, it is easy to understand that the English auction achieves the same outcome as the second-price sealed-bid auction does. For the assignment market of Koopmans and Beckmann (1957) and Shapley and Shubik (1971), Crawford and Knoer (1981) proposed the first dynamic auction which converges to a competitive equilibrium by a limiting argument. Leonard (1983) showed that the minimum competitive equilibrium price vector of this market coincides with the VCG payment. Demange et al. (1986) proved that their dynamic auction finds the minimum competitive equilibrium price vector and is strategy-proof in the sense of achieving an *ex post Nash equilibrium* for sincere bidding. For the general gross substitutes (GS) model of Kelso and Crawford (1982), Gul and Stacchetti (2000) demonstrated that their dynamic auction finds the minimum competitive equilibrium price vector but cannot be strategy-proof. Ausubel (2006) proposed a strategy-proof dynamic auction for the GS model by ingeniously exploring the $m+1$ markets $\mathcal{M}_{-j}$ for $j \in B_0$ in the definition of the VCG outcome. His analysis (Ausubel 2006, pp. 612-616, 622-624) on the strategy-proof outcome concentrates on *divisible goods* and relies on calculus, convex analysis, and Theorem 1 of Krishna and Maenner (2001). After introducing his basic dynamic auction for the indivisible GS goods, Ausubel (2006, p. 620) briefly mentioned that his argument on strategy-proof results for the divisible goods also applies to the indivisible GS case.

Here we offer a general analysis on strategic issues concerning *indivisible goods and all unimodular demand types.* Although Ausubel's auction and ours share similar strategic properties such as *ex post perfect Nash equilibrium for sincere bidding*, his analysis and ours are markedly different in nature and complement each other. Our analysis has to use combinatorial arguments and rely on recent progress in discrete/combinatorial optimization. More precisely, our strategy-proof results Theorems 4 and 5 depend on Theorem 3 in Section 4 which in turn depends on Theorems 1 and 2, whose proofs rely on recent results from discrete optimization, quite distinct from calculus and convex analysis. Recall that Assumptions (A1) and (A2) for our model underlie our results of discrete/combinatorial nature. Barring the use of Theorem 3, the argument for our Theorems 4 and 5 is combinatorial, intuitive, elementary, and noticeably different from that of Ausubel (2006).

5.1 Incentive Compatible Dynamic Auction Design

We now introduce an incentive-compatible dynamic auction mechanism based on the UCD auction. Because bidders are strategic agents, they may submit whatever bids they like in their best interests without openly flouting the auction rules and therefore their bids could be different from their true demand sets. The mechanism runs the UCD auction as described in Section 4 for every market $\mathcal{M}_{-k}$ ($k \in B_0$) with the following modifications. Consider every market $\mathcal{M}_{-k}$, $k \in B_0$. Let $p^k(t) \in \mathbb{Z}^N$ denote the prices of the market $\mathcal{M}_{-k}$ at time $t \in \mathbb{Z}_+$. Then at time $t \in \mathbb{Z}_+$ and with respect to $p^k(t)$, every bidder $j \in B_{-k}$ submits a bid $B^j_k(t) \subseteq \{0,1\}^N$ which may differ from his true demand set $D^j(p^k(t))$, but *the seller's bid* $B^0_k(t)$ *always equals her true demand set* $D^0(p^k(t))$. The auctioneer calculates an optimal solution $\delta^k(t)$ to the following modified righthand problem of (14)

$$\max_{\delta \in \mathcal{SD}} \{ \sum_{j \in B_{-k}} \min_{x^j \in B_k^j(t)} x^j \cdot \delta - \sum_{i \in N} \delta_i \}. \tag{15}$$

It is important to observe that in the above formula we replace the true demand set $D^j(p^k(t))$ in (14) by bid $B_k^j(t)$ in order to take strategic behavior of bidders into consideration. This can change the outcome of the auction and may have serious implications.

When the vector $\mathbf{0}$ of zeros is an optimal solution to (15), this means that the auction finds an 'equilibrium allocation' $X^k = (x^{k,j}, j \in B_{-k})$ in the market $\mathcal{M}_{-k}$ in the sense that $x^{k,j} \in B_k^j(t)$ for every $j \in B_{-k}$ and $\sum_{j \in B_{-k}} x^{k,j} = \sum_{i \in N} e(i)$. Otherwise, when $\mathbf{0}$ is not an optimal solution to (15), the auctioneer updates prices by setting $p^k(t+1) = p^k(t) + \delta^k(t)$. Because bidders may act strategically and so their bids may not be their true demand sets, it is possible that the auction may never find an equilibrium allocation in some market $\mathcal{M}_{-k}$. In this case, the auction fails to terminate and will require every bidder to pay a penalty $c > 0$ for nothing. We now present the auction.

The Incentive Compatible Universal Dynamic (ICUD) Auction

Step 1: At first, the auctioneer announces a common price vector $p^k(0) = p(0) \in \mathbb{Z}^N$ for all markets $\mathcal{M}_{-k}$, $k \in B_0$. Let $t := 0$ and go to **Step 2**.

Step 2: At prices $p^k(t) \in \mathbb{Z}^N$, every agent $j \in B_{-k}$ submits her bid $B_k^j(t) \subseteq \{0,1\}^N$. Based on the reported bids, if the vector $\mathbf{0}$ of zeros is an optimal solution to (15), the auctioneer finds an equilibrium allocation X^k in market $\mathcal{M}_{-k}$, and records the current prices as $p^k(T^k) \in \mathbb{Z}^n$ and the current time as $T^k \in \mathbb{Z}_+$. For any market $\mathcal{M}_{-k}$ which is not 'in equilibrium', the auctioneer calculates an optimal solution $\delta^k(t)$ to (15) and announces a new price vector $p^k(t+1) = p^k(t) + \delta^k(t)$. The auction goes back to **Step 2** with $t := t + 1$. If the auction has found an equilibrium allocation X^k in every market $\mathcal{M}_{-k}$, $k \in B_0$, go to **Step 3**.

Step 3: All markets now clear. For every $k \in B_0$ and every agent $j \in B_{-k}$ at every time $t = 0, 1, \cdots, T^k - 1$, based on her reported bids $B_k^j(t)$ and the price change $\delta^k(t)$, the auctioneer calculates agent j's 'indirect utility reduction' $\Delta_j^k(t)$ when prices are changed at time t from $p^k(t)$ to $p^k(t+1)$ in the market $\mathcal{M}_{-k}$, where

$$\Delta_j^k(t) = \min_{x^j \in B_k^j(t)} x^j \cdot \delta^k(t). \tag{16}$$

Every bidder $j \in B$ will be assigned the bundle $x^{0,j}$ of the allocation $X^0 = (x^{0,j}, j \in B_0)$ found in the original market $\mathcal{M}_{-0} = \mathcal{M}$ and asked to pay β_j, with the option to decline and walk away freely, when his payoff becomes negative, where

$$\beta_j = \sum_{h \in B_{-j}} \Big[\Big(\sum_{t=0}^{T^0-1} \Delta_h^0(t) - \sum_{t=0}^{T^j-1} \Delta_h^j(t) \Big) + x^{j,h} \cdot p^j(T^j) - x^{0,h} \cdot p^0(T^0) \Big]. \tag{17}$$

The seller keeps the bundle $x^{0,0}$ of the allocation X^0 and receives the total payment $\sum_{j \in B} \beta_j$. The auction stops.

The payment formula (17) has three terms and can be explained intuitively as follows: The first term is the accumulation of 'indirect utility reduction' of bidder j's opponents in B_{-j} along the path from $p^j(T^j)$ to $p(0)$ in the market $\mathcal{M}_{-j}$ and along the path from $p(0)$ to $p^0(T^0)$ in the market $\mathcal{M}$; the second term stands for the total equilibrium payment by all bidders in the market $\mathcal{M}_{-j}$, i.e., all opponents of bidder j; and the third term represents the total equilibrium payment by all opponents of bidder j in the market $\mathcal{M}$. The final payment β_j of bidder j equals the first term by adding the second term and subtracting the third term. This payment formula is simple and easy to calculate, using only revealed information, and having an intimate relation with the VCG payment as to be shown later. Every bidder can easily use this payment formula to calculate his own payment so can the seller for every bidder.

A bidder j is said to *make mistakes or manipulate* if his bid $B^j(t)$ does not equal his true demand set $D^j(p(t))$ at prices $p(t)$. Observe that in Step 3 of our auction we allow any bidder $j \in B$ to decline any unacceptable assignment and walk away freely, if accepting the assignment would give him a negative utility of $u^j(x^{0,j}) - \beta_j < 0$, which is caused by mistakes or manipulation. We call this option of letting bidders walk away empty-handed without paying any penalty *a lenient policy.* This lenient policy is different from Ausubel's. In his auction, no bidder is given any opportunity to walk away freely and may have to pay a huge amount[6] according to the payment formula (7) of Ausubel (2006, p. 611) if mistakes or manipulation have been made before a time $\bar{t}$. This ends our discussion on the case when the auction terminates in Step 3, i.e., in finite time. Now we turn to another case–*the broken down case*–when the auction does not terminate. In this case, our auction adopts *a slightly different lenient policy* which requires every bidder

[6]This amount depends on bidders' behavior and is not known in advance.

to pay a fixed penalty $c > 0$ and get no item. This lenient policy is also different from Ausubel's auction (2006, p. 613) which imposes a severe penalty of infinity.

These policies are not so innocent as they might appear. For Ausubel's auction, the severe penalty of infinity is necessary for the broken down case, because when bidders make mistakes or manipulate and his auction stops, the payment of every bidder can be extremely large and is unknown in advance so the only way of preventing his auction from not stopping is to impose the penalty of infinity for every bidder. For our auction, the light penalty of $c > 0$ for the broken case is possible, because our auction in Step 3 allows bidders to walk away freely if their payoffs become negative. Our lenient policies provide better opportunities for buyers to learn and adjust without paying high costs. But they could be a disadvantage to the seller in the sense that the seller might not get a high penalty as given by Ausubel's auction.

It is also interesting to note that our ICUD auction can tolerate any mistake or manipulation made by bidders and allows bidders to learn, adjust, and correct so that for any time $t^* \in \mathbb{Z}_+$, no matter what has happened before t^*, as long as from t^* on every bidder bids truthfully and Assumptions (A1) and (A2) are satisfied, the ICUD auction will find a competitive equilibrium in every market in finite time in Step 3, because the UCD auction converges to a competitive equilibrium wherever it starts from $\mathbb{Z}^N$. In this case, bidders may have to pay more by (17) than they act honestly and make no mistakes. But they will never pay to such an extent that their payoffs become negative.

Another difference between Ausubel's strategy-proof auction and our ICUD auction is that his auction and payment rules are not symmetric and payment formula (7) of Ausubel (2006, p.611) involves Stieltjes integrals of continuous price functions, whereas our ICUD auction and payment rules (16) and (17) are symmetric, simple, and easy to calculate.[7]

To facilitate a better understanding of the ICUD auction we use Example 1 to illustrate its operation before investigating its strategic properties. The auction starts with the prices

[7] Our ICUD auction starts with the same initial price vector $p(0)$ for all markets $\mathcal{M}$ and $\mathcal{M}_{-j}$, $j \in B$, whereas Ausubel's auction (Ausubel 2006, pp. 615-616) starts with the same initial price vector $p(0)$ only for the markets $\mathcal{M}_{-j}$, $j \in B$, but for the market $\mathcal{M}$ his auction starts with the equilibrium price vector p^{-k^*} of any chosen market $\mathcal{M}_{-k^*}$. In his auction, the payment of bidder k^* is given by Equation (7) (Ausubel 2006, p. 611) using the price vectors along the path from p^{-k^*} to p^*. The payment of bidder j ($j \in B_{-k^*}$) is also given by Equation (7) but using the price vectors along the path from p^{-j} to p^0; the path from p^0 to p^{-k^*}; and the path from p^{-k^*} to p^*.

$p(0) = (p_A(0), p_B(0)) = (0, 0)$ and terminates in round $t = 5$. Prices $p^k(t)$, increments $\delta^k(t)$, bids $B_k^j(t)$, and indirect utility reductions $\Delta_j^k(t)$ in each round t are shown in Table 4. At $t = 5$, we have $p^0(5) = p^1(5) = p^2(5) = p^3(5) = (2, 3)$, $X^0 = (x^{0,0}, x^{0,1}, x^{0,2}, x^{0,3}) = (\emptyset, AB, \emptyset, \emptyset)$, $X^1 = (x^{1,0}, x^{1,2}, x^{1,3}) = (\emptyset, AB, \emptyset)$, $X^2 = (x^{2,0}, x^{2,1}, x^{2,3}) = (\emptyset, AB, \emptyset)$, and $X^3 = (x^{3,0}, x^{3,1}, x^{3,2}) = (\emptyset, \emptyset, AB)$. We also have $B_{-1} = \{0, 2, 3\}$, $B_{-2} = \{0, 1, 3\}$, and $B_{-3} = \{0, 1, 2\}$. Taking utility reductions $\Delta_j^k(t)$ in Table 4 into pricing formula (17) yields $\beta_1 = 5$, $\beta_2 = 0$ and $\beta_3 = 0$. Here we give one instance in detail:

$$
\begin{aligned}
\beta_1 &= \sum_{t=0}^4 \Delta_0^0(t) - \sum_{t=0}^4 \Delta_0^1(t) + x^{1,0} \cdot p^1(5) - x^{0,0} \cdot p^0(5) \\
&+ \sum_{t=0}^4 \Delta_2^0(t) - \sum_{t=0}^4 \Delta_2^1(t) + x^{1,2} \cdot p^1(5) - x^{0,2} \cdot p^0(5) \\
&+ \sum_{t=0}^4 \Delta_3^0(t) - \sum_{t=0}^4 \Delta_3^1(t) + x^{1,3} \cdot p^1(5) - x^{0,3} \cdot p^0(5) = 5
\end{aligned}
$$

Consequently, bidder 1 gets the bundle AB and pays 5 in return and the other two bidders get nothing and pay nothing. The seller receives 5 for the sale of her goods A and B.

5.2 The Dynamic Auction Game and Its Strategic Properties

Now we discuss how the ICUD auction can induce strategic bidders to bid truthfully as price-takers, generating efficient outcomes even when these bidders have market power. In particular, we will show that sincere bidding is an ex post perfect Nash equilibrium. This can be seen as a vivid practical application of the fundamental solution concept for dynamic games of incomplete information; see Fudenberg and Tirole (1991).

We need to formulate our ICUD auction as an extensive-form dynamic game of incomplete information. In this (dynamic) auction game, all bidders are players. Prior to the start of the game, every player $j \in B$ knows privately only his own value function u^j satisfying Assumptions (A1) and (A2). The auctioneer knows that every bidder's utility function satisfies Assumptions (A1) and (A2) but does not know their utility functions. The auctioneer initially announces a common price vector for all markets and every bidder responds by reporting his bid to the auctioneer for every market in which he is involved. Then based on reported bids the auctioneer checks if the aggregated demands equal the aggregated supplies in every market or not. If all markets are cleared, the auction stops. Otherwise, the auctioneer adjusts prices and bidders update their bids.

Table 4: Illustration of the ICUD Auction.

Time t	Prices $P^k(t)$	Increments $\delta^k(t)$	Bids $B_k^j(t)$	Utility Reductions $\Delta_j^k(t)$
$t=0$	$P^0(0)=(0,0)$ $P^1(0)=(0,0)$ $P^2(0)=(0,0)$ $P^3(0)=(0,0)$	$\delta^0(0)=(1,0)$ $\delta^1(0)=(1,0)$ $\delta^2(0)=(1,0)$ $\delta^3(0)=(1,0)$	$B_0^0(0)=\{AB\}$, $B_0^1(0)=\{AB\}$ $B_0^2(0)=\{AB\}$, $B_0^3(0)=\{AB\}$ $B_1^0(0)=\{AB\}$, $B_1^2(0)=\{AB\}$, $B_1^3(0)=\{AB\}$ $B_2^0(0)=\{AB\}$, $B_2^1(0)=\{AB\}$, $B_2^3(0)=\{AB\}$ $B_3^0(0)=\{AB\}$, $B_3^1(0)=\{AB\}$, $B_3^2(0)=\{AB\}$	$\Delta_0^0(0)=1$, $\Delta_1^0(0)=1$, $\Delta_2^0(0)=1$, $\Delta_3^0(0)=1$ $\Delta_0^1(0)=1$, $\Delta_2^1(0)=1$, $\Delta_3^1(0)=1$ $\Delta_0^2(0)=1$, $\Delta_1^2(0)=1$, $\Delta_3^2(0)=1$ $\Delta_0^3(0)=1$, $\Delta_1^3(0)=1$, $\Delta_2^3(0)=1$
$t=1$	$P^0(1)=(1,0)$ $P^1(1)=(1,0)$ $P^2(1)=(1,0)$ $P^3(1)=(1,0)$	$\delta^0(1)=(1,0)$ $\delta^1(1)=(1,0)$ $\delta^2(1)=(1,0)$ $\delta^3(1)=(1,0)$	$B_0^0(1)=\{AB\}$, $B_0^1(1)=\{AB\}$ $B_0^2(1)=\{AB\}$, $B_0^3(1)=\{AB\}$ $B_1^0(1)=\{AB\}$, $B_1^2(1)=\{AB\}$, $B_1^3(1)=\{AB\}$ $B_2^0(1)=\{AB\}$, $B_2^1(1)=\{AB\}$, $B_2^3(1)=\{AB\}$ $B_3^0(1)=\{AB\}$, $B_3^1(1)=\{AB\}$, $B_3^2(1)=\{AB\}$	$\Delta_0^0(1)=1$, $\Delta_1^0(1)=1$, $\Delta_2^0(1)=1$, $\Delta_3^0(1)=1$ $\Delta_0^1(1)=1$, $\Delta_2^1(1)=1$, $\Delta_3^1(1)=1$ $\Delta_0^2(1)=1$, $\Delta_1^2(1)=1$, $\Delta_3^2(1)=1$ $\Delta_0^3(1)=1$, $\Delta_1^3(1)=1$, $\Delta_2^3(1)=1$
$t=2$	$P^0(2)=(2,0)$ $P^1(2)=(2,0)$ $P^2(2)=(2,0)$ $P^3(2)=(2,0)$	$\delta^0(2)=(0,1)$ $\delta^1(2)=(0,1)$ $\delta^2(2)=(0,1)$ $\delta^3(2)=(0,1)$	$B_0^0(2)=\{AB,B\}$, $B_0^1(2)=\{AB\}$ $B_0^2(2)=\{AB\}$, $B_0^3(2)=\{AB\}$ $B_1^0(2)=\{AB,B\}$, $B_1^2(2)=\{AB\}$, $B_1^3(2)=\{AB\}$ $B_2^0(2)=\{AB,B\}$, $B_2^1(2)=\{AB\}$, $B_2^3(2)=\{AB\}$ $B_3^0(2)=\{AB,B\}$, $B_3^1(2)=\{AB\}$, $B_3^2(2)=\{AB\}$	$\Delta_0^0(2)=1$, $\Delta_1^0(2)=1$, $\Delta_2^0(2)=1$, $\Delta_3^0(2)=1$ $\Delta_0^1(2)=1$, $\Delta_2^1(2)=1$, $\Delta_3^1(2)=1$ $\Delta_0^2(2)=1$, $\Delta_1^2(2)=1$, $\Delta_3^2(2)=1$ $\Delta_0^3(2)=1$, $\Delta_1^3(2)=1$, $\Delta_2^3(2)=1$
$t=3$	$P^0(3)=(2,1)$ $P^1(3)=(2,1)$ $P^2(3)=(2,1)$ $P^3(3)=(2,1)$	$\delta^0(3)=(0,1)$ $\delta^1(3)=(0,1)$ $\delta^2(3)=(0,1)$ $\delta^3(3)=(0,1)$	$B_0^0(3)=\{AB,B,\emptyset\}$, $B_0^1(3)=\{AB\}$ $B_0^2(3)=\{AB\}$, $B_0^3(3)=\{AB\}$ $B_1^0(3)=\{AB,B,\emptyset\}$, $B_1^2(3)=\{AB\}$, $B_1^3(3)=\{AB\}$ $B_2^0(3)=\{AB,B,\emptyset\}$, $B_2^1(3)=\{AB\}$, $B_2^3(3)=\{AB\}$ $B_3^0(1)=\{AB,B,\emptyset\}$, $B_3^1(3)=\{AB\}$, $B_3^2(3)=\{AB\}$	$\Delta_0^0(3)=0$, $\Delta_1^0(3)=1$, $\Delta_2^0(3)=1$, $\Delta_3^0(3)=1$ $\Delta_0^1(3)=0$, $\Delta_2^1(3)=1$, $\Delta_3^1(3)=1$ $\Delta_0^2(3)=0$, $\Delta_1^2(3)=1$, $\Delta_3^2(3)=1$ $\Delta_0^3(3)=0$, $\Delta_1^3(3)=1$, $\Delta_2^3(3)=1$
$t=4$	$P^0(4)=(2,2)$ $P^1(4)=(2,2)$ $P^2(4)=(2,2)$ $P^3(4)=(2,2)$	$\delta^0(4)=(0,1)$ $\delta^1(4)=(0,1)$ $\delta^2(4)=(0,1)$ $\delta^3(4)=(0,1)$	$B_0^0(4)=\{\emptyset\}$, $B_0^1(4)=\{AB\}$ $B_0^2(4)=\{\emptyset\}$, $B_0^3(4)=\{AB,\emptyset\}$ $B_1^0(4)=\{\emptyset\}$, $B_1^2(4)=\{AB\}$, $B_1^3(4)=\{AB,\emptyset\}$ $B_2^0(4)=\{\emptyset\}$, $B_2^1(4)=\{AB\}$, $B_2^3(4)=\{AB,\emptyset\}$ $B_3^0(4)=\{\emptyset\}$, $B_3^1(4)=\{AB\}$, $B_3^2(4)=\{AB\}$	$\Delta_0^0(4)=0$, $\Delta_1^0(4)=1$, $\Delta_2^0(4)=1$, $\Delta_3^0(4)=0$ $\Delta_0^1(4)=0$, $\Delta_2^1(4)=1$, $\Delta_3^1(4)=0$ $\Delta_0^2(4)=0$, $\Delta_1^2(4)=1$, $\Delta_3^2(4)=0$ $\Delta_0^3(4)=0$, $\Delta_1^3(4)=1$, $\Delta_2^3(4)=1$
$t=5$	$P^0(5)=(2,3)$ $P^1(5)=(2,3)$ $P^2(5)=(2,3)$ $P^3(5)=(2,3)$	$\delta^0(5)=(0,0)$ $\delta^1(5)=(0,0)$ $\delta^2(5)=(0,0)$ $\delta^3(5)=(0,0)$	$B_0^0(5)=\{\emptyset\}$, $B_0^1(5)=\{AB,A,\emptyset\}$ $B_0^2(5)=\{AB,A,\emptyset\}$, $B_0^3(5)=\{\emptyset\}$ $B_1^0(5)=\{\emptyset\}$, $B_1^2(5)=\{AB,A,\emptyset\}$, $B_1^3(5)=\{\emptyset\}$ $B_2^0(5)=\{\emptyset\}$, $B_2^1(5)=\{AB,A,\emptyset\}$, $B_2^3(5)=\{\emptyset\}$ $B_3^0(5)=\{\emptyset\}$, $B_3^1(5)=\{AB,A,\emptyset\}$, $B_3^2(5)=\{AB,A,\emptyset\}$	$\Delta_0^0(5)=0$, $\Delta_1^0(5)=0$, $\Delta_2^0(5)=0$, $\Delta_3^0(5)=0$ $\Delta_0^1(5)=0$, $\Delta_2^1(5)=0$, $\Delta_3^1(5)=0$ $\Delta_0^2(5)=0$, $\Delta_1^2(5)=0$, $\Delta_3^2(5)=0$ $\Delta_0^3(5)=0$, $\Delta_1^3(5)=0$, $\Delta_2^3(5)=0$

In this auction, announced prices in each market can be observed by all bidders. Every bidder knows of course his own bids. Whether a bidder can observe bids of other bidders depends on the specification of the auction rule. In the current auction the auctioneer can ask every bidder to either publicly reveal his bids or just submit his bids privately to her. We use H_j^t to denote the part of the information or history of play that player j has observed so far right after prices at time $t \in \mathbb{Z}_+$ have been announced but no players have placed their bids at the current prices. A natural specification is that H_j^t contains his own utility function u^j, all observable prices before and at time t in every market in which he takes part, all his own bids and all possibly revealed bids of other players before time t.

At every time $t \in \mathbb{Z}_+$, after the auctioneer announces current prices for each market, every bidder will think about how to bid based upon all currently available information to him. The (dynamic) *strategy* σ_j of player j, $j \in B$, is a set-valued function which specifies his bids $\sigma_j(t, k, H_j^t) = B_k^j(t) \subseteq \{0, 1\}^N$ for every market $\mathcal{M}_{-k}$, $k \in B_0 \setminus \{j\}$, at every time $t \in \mathbb{Z}_+$, and for every history H_j^t. Let Σ_j denote the strategy space of all player j's strategies σ_j. Obviously, player j's strategy space Σ_j contains his sincere bidding strategies as specified in Definition 7 and many other strategies as well. The outcome of the ICUD auction game relies totally upon the auction rules, the histories, and the strategies the bidders may adopt. When every bidder $j \in B$ takes a strategy $\sigma_j \in \Sigma_j$ and the ICUD auction terminates in Step 3, then bidder $j \in B$ receives bundle $x^{0,j}$ and pays β_j given by (17), or simply walks away. In this case, his payoff equals $\max\{u^j(x^{0,j}) - \beta_j, 0\}$. Otherwise, the auction is in the broken down state in which every bidder gets no item but pays a fixed penalty $c > 0$.

In the literature for static auction games of incomplete information, the notion of *ex post equilibrium* has been used by Cremér and McLean (1985), Krishna (2002), and Perry and Reny (2005). This solution requires that the strategy for every player should remain optimal if the player were to get to know types of his opponents. Ausubel (2004, 2006) and Sun and Yang (2014) have adopted the solution of *ex post perfect equilibrium* to dynamic auction games of incomplete information which requires the same condition for every player at every node of the dynamic auction game.

Definition 10 (Ex Post Perfect Nash Equilibrium) The strategy m-tuple $\{\sigma_j\}_{j\in B}$ of the dynamic auction game of incomplete information is *an ex post perfect* (*Nash*) *equilibrium* if for every time $t \in \mathbb{Z}_+$, following any history $\{H_j^t\}_{j\in B}$, and for any realization $\{u^j\}_{j\in B}$ of private information, the continuation strategies $\sigma_j(\cdot, \cdot, \cdot \mid t, k, H_j^t)$ for every player $j \in B$ and for every market $k \in B_0 \setminus \{j\}$ constitute a Nash equilibrium of the game even if the realization $\{u^j\}_{j\in B}$ becomes common knowledge.

An important advantage of ex post perfect equilibrium over Bayesian equilibrium or perfect Bayesian equilibrium is that it is not only robust against any regret but also independent of any probability distribution. It is very useful in practice as it is very difficult to elicit or gauge a probability distribution of a bidder's valuation. The notion of ex post perfect equilibrium is a refinement of ex post Nash equilibrium and therefore more desirable and stronger than the latter.

Now we demonstrate several important and appealing properties of the ICUD auction.

Theorem 4 *Under Assumptions* (A1) and (A2)*, if every bidder bids sincerely, the ICUD auction converges to a competitive equilibrium, yielding a VCG outcome for the market* $\mathcal{M}$*, in a finite number of rounds.*

Theorem 5 *Under Assumptions* (A1) and (A2)*, sincere bidding by every bidder is an ex post perfect equilibrium in the ICUD auction.*

We say that a mechanism is *beneficial to every agent* if the payment the seller receives for every sold bundle is at least as big as her reserve price of the bundle or the total utility she receives is at least as good as she does not trade, and if the net profit for every bidder is nonnegative.

Proposition 4 *Under Assumptions* (A1) and (A2)*, if every bidder bids sincerely, the ICUD auction mechanism is beneficial to every agent, provided that the seller's utility function* u^0 *is either submodular or superadditive.*

An auction mechanism is said to be *ex post individually rational*, if, for every bidder, no matter how his opposing bidders act in the auction, as long as he is sufficiently able to judge whether his payoff is negative or nonnegative, he will never end up with a negative payoff. This property is quite desirable for practical auction design. We conclude with the following proposition.

Proposition 5 *Under Assumptions* (A1) and (A2)*, the ICUD auction is ex post individually rational.*

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Appendix

We first review several mathematical concepts. A polytope can be also defined as a convex hull of finitely many vectors in $\mathbb{R}^N$. An *edge* of a polyhedron is a face of dimension one and a vector is an *edge-direction vector* of an edge if it is a non-zero scalar multiple of the difference of any two distinct points on the edge. So, if v is an edge-direction vector, then αv for any $\alpha \neq 0$ is also an edge-direction vector so is $-\alpha v$.

A set $S \subseteq \mathbb{Z}^N$ is *discrete convex* if $S = \text{Conv}(S) \cap \mathbb{Z}^N$. A function $f : \mathbb{Z}^N \to \mathbb{R}$ is *discrete concave* if, for any finite number of $\lambda_j \geq 0$, $j = 1, \cdots, t$ and any $x^j \in \mathbb{Z}^N$ for $j = 1, \cdots, t$ with $\sum_{j=1}^t \lambda_j = 1$ and $\sum_{j=1}^t \lambda_j x^j \in \mathbb{Z}^N$, we have $f(\sum_{j=1}^t \lambda_j x^j) \geq \sum_{j=1} \lambda_j f(x^j)$. Given a lattice $S \subseteq \mathbb{Z}^N$, a function $f : S \to \mathbb{R} \cup \{+\infty\}$ is *submodular* if $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ for any $x, y \in S$. When a utility function of items is submodular, it has decreasing marginal returns over any item. This means that items exhibits substitutability. A function $f : S \to \mathbb{R}$ is *subadditive* if $f(x+y) \leq f(x)+f(y)$ for any $x, y \in S$. Subadditivity reflects a more general substitutability. A function f is *supermodular* if $-f$ is submodular. If a utility function of items is supermodular, then these items have increasing marginal returns and show complementarity. A function f is *superadditive* if $-f$ is subadditive. Superadditivity is more general than supermodularity. See Murota (2003) and Fujishige (2005) in detail. Given a utility function $u : S \to \mathbb{R} \cup \{-\infty\}$ with a finite set $S \subset \mathbb{Z}^N$ and $\sharp\text{dom}(u) > 1$, we say that a set $\mathcal{D}_u$ of primitive vectors in $\mathbb{Z}^N$ is a *demand edge-set* of function u if every $v \in \mathcal{D}_u$ is an edge-direction vector of the convex hull of some demand set $D_u(p)$ with $\sharp D_u(p) > 1$. Observe that we have $v \in \mathcal{D}_u$ if and only if v is normal to some facet of the LIP $\mathcal{T}_u$.

Proof of Proposition 1: Let $\mathcal{D} = \{v^1, v^2, \cdots, v^k, -v^1, -v^2, \cdots, -v^k\}$ be an arbitrarily given unimodular demand type. To prove the result, it suffices to consider the $n \times k$ integer matrix $A = [v^1, v^2, \cdots, v^k]$. If the rank of A, denoted by rank(A), is equal to n, we are done. Assume that rank$(A) = r < n$. Choose a submatrix B formed by r linearly independent columns of A. Then by definition there exists an $n \times (n - r)$ matrix C such that the matrix $U \equiv [C \vdots B]$ is unimodular. Below I_l and $O_{(n-l)\times l}$ represent the identity matrix of order l and an $(n - l) \times l$ matrix with entries $0's$, respectively. We have

$$U^{-1}[C \vdots B] = \begin{pmatrix} I_{n-r} & O_{(n-r)\times r} \\ O_{r\times(n-r)} & I_r \end{pmatrix} \tag{18}$$

Now choose another submatrix D formed by r linearly independent columns of A. To ease exposition but without loss of generality, we assume that $A = [B\vdots D]$. We will show that $[C\vdots D]$ is unimodular, i.e., both B and D can use the common set C to form unimodular matrices. Note that for some $r \times r$ matrix D_0 with $\text{rank}(D_0) = r$ we have

$$U^{-1}[C\vdots B\vdots D] = \begin{pmatrix} I_{n-r} & O_{(n-r)\times r} & O_{(n-r)\times r} \\ O_{r\times(n-r)} & I_r & D_0 \end{pmatrix}, \tag{19}$$

where the form of the last r columns follows from the assumption that $\text{rank}(A) = r$.

Moreover, there exists an $n \times (n - r)$ matrix E such that the matrix $V \equiv [E\vdots D]$ is unimodular. Then for some $r \times r$ matrix B_0 with $\text{rank}(B_0) = r$ we have

$$V^{-1}[C\vdots B\vdots D] = \begin{pmatrix} \tilde{C}_1 & O_{(n-r)\times r} & O_{(n-r)\times r} \\ \tilde{C}_2 & B_0 & I_r \end{pmatrix}. \tag{20}$$

Since U and V are unimodular, it follows from (20) that we have

$$\det(V^{-1}[C\vdots B]) = \det(\tilde{C}_1)\det(B_0) = \pm 1 (= \det(V^{-1}U)). \tag{21}$$

Hence from (21) we have

$$\det(B_0) = \pm 1. \tag{22}$$

since $\tilde{C}_1$ and B_0 are integer matrices.

Because of the symmetry between (B, U) and (D, V) we also have

$$\det(D_0) = \pm 1 \tag{23}$$

as a counterpart of (22). Hence, $[C\vdots D]$ is also unimodular because of (19). Consequently, we can use C instead of E for D to get a unimodular matrix $[C\vdots D]$. We are done. □

The proof of Lemma 1 is easy. Also, Lemma 2 follows immediately from the definition of Lyapunov function $\mathcal{L}$ and Lemma 1.

Proof of Lemma 3: By the assumption the demand edge-set $\mathcal{D}_u$ is full-dimensional and so is the demand type $\mathcal{D}(\supseteq \mathcal{D}_u)$. Let x^* be an extreme point of the full-dimensional, convex hull of the set $D_u(p)$. There exists a set of n linearly independent edge-direction vectors $d_1, \cdots, d_n \in \mathcal{D}_u$ that are extreme vectors of the tangent cone of the convex hull of the set $D_u(p)$ at x^*. Let $y = p \cdot (x - x^*) + u(x^*)$ be the hyperplane that supports u at every point of $D_u(p)$. Then we have

$$p \cdot d_i = u(x^* + d_i) - u(x^*) \qquad (\forall i = 1, \cdots, n). \tag{24}$$

Since $d_1, \cdots, d_n \in \mathcal{D}$ form a unimodular matrix and the right-hand side of (24) is an integer for each $i = 1, \cdots, n$ by the assumption, p is a unique integral vector satisfying equation (24). □

The following result is given in Tran and Yu (2019), revealing an important property concerning the unimodular demand type and will be used in our proof of Lemma 4 below. See Murota and Tamura (2024) for a survey on this result.

Lemma 6 *Suppose that M is a unimodular matrix, and that P and Q are integral polytopes with edges parallel to columns of M. Then, $P \cap \mathbb{Z}^N + Q \cap \mathbb{Z}^N = (P + Q) \cap \mathbb{Z}^N$.*

Proof of Lemma 4: Take any $x^j \in D^j(p)$ for all $j \in B_0$. Then $g(x) = \sum_{j \in B_0} u^j(x^j)$ with $x = \sum_{j \in B_0} x^j$. By definition for all $j \in B_0$ we have

$$u^j(x^j) - p \cdot x^j \geq u^j(y^j) - p \cdot y^j, \text{ for all } y^j \in \mathrm{dom}(g). \tag{25}$$

Clearly, for all $y^j \in \mathrm{dom}(g)$ $(j \in B_0)$ satisfying $\sum_{j \in B_0} x^j = \sum_{j \in B_0} y^j$ we have

$$u^j(x^j) - p \cdot x^j \geq u^j(y^j) - p \cdot y^j.$$

Now adding all inequalities up yields

$$\sum_{j \in B_0} u^j(x^j) \geq \sum_{j \in B_0} u^j(y^j) \tag{26}$$

for all $y^j \in \mathrm{dom}(g)$ $(j \in B_0)$ satisfying $\sum_{j \in B_0} x^j = \sum_{j \in B_0} y^j$. By definition $u(x) = \sum_{j \in B_0} u^j(x^j)$. Observe that the inequality (26) still holds true if $g(z) = \sum_{j \in B_0} u^j(z^j)$ with $x = z = \sum_{j \in B_0} z^j$ and $z^j \in D^j(q)$ for $j \in B_0$ and $q \neq p$. This shows $g(x) = u(x)$ and g is well-defined.

Because of the definition of convolution, for any $p \in \mathbb{R}^N$ we have

$$D_u(p) = \max\{u(x) - p \cdot x \mid x \in \mathbb{Z}^N\} = \sum_{j \in B_0} \max\{u(x^j) - p \cdot x^j \mid x^j \in \{0,1\}^N\}. \tag{27}$$

It is clear that $D_u(p) = D^{Ms}(p)$. Hence we have the following relation, i.e., equation (6):

$$D^{Ms}(p) = D^0(p) + D^1(p) + \cdots + D^m(p) = D_u(p). \tag{28}$$

Since by Assumption (A2) all $D^j(p)$ $(j \in B_0)$ have the same unimodular demand type $\mathcal{D}$ and the Minkowski-sum operation is associative, it follows from (28) and Lemma 6 that

$D_u(p)$ also has the same unimodular demand type $\mathcal{D}$ and $\mathrm{Conv}(D_u(p)) \cap \mathbb{Z}^N = D_u(p)$. Because for every $p \in \mathbb{R}^N$ the set $D_u(p)$ is discrete convex, u is clearly a discrete concave function with the unimodular demand type $\mathcal{D}$. □

Proof of Theorem 1: Let P be the set of competitive equilibrium price vectors. It follows from Baldwin and Klemperer (2018, Theorem 4.3) that there exists at least one competitive equilibrium price vector. Because all u^j, $j \in B_0$, are integer-valued and of unimodular demand type $\mathcal{D}$, it follows from Lemma 4 that their convolution u is a discrete concave integer-valued function with the same unimodular demand type $\mathcal{D}$. We know that $p \in \mathbb{R}^N$ is a competitive equilibrium price vector if and only if it is a minimizer of the Lyapunov function $\mathcal{L}$. The convexity of the function $\mathcal{L}$ implies that the set P is a polyhedral convex set since the function $\mathcal{L}$ is polyhedral by Lemma 2. Clearly, it is nonempty and bounded, and hence it is a polytope.

Next we prove that every vertex of P is integral. This follows immediately from the fact that the extreme points of the set P are normal vectors p of hyperplanes supporting the convolution u at a full-dimensional demand set $D^{Ms}(p)$ and hence integral by Lemma 3 because of Assumptions (A1) and (A2). □

Proof of Lemma 5: Let $M = [d_1, \cdots, d_{n-1}, d_n]$ be the $n \times n$ matrix and δ^* be the nth row of M^{-1}. Then we have $\delta^* \cdot d_j = 0$ for $j = 1, \cdots, n-1$ and $\delta^* \cdot d_n = 1$. Since M is a unimodular matrix, δ^* is an integral vector. Hence $\delta^* = \alpha\delta$ or $\delta^* = -\alpha\delta$ for some $\alpha \geq 1$ because of the definition of δ. Consequently, we have $\alpha|\delta \cdot d_n| = \delta^* \cdot d_n = 1$. □

Proof of Proposition 2: We only need to consider the case that $p(t)$ is not a Walrasian equilibrium price vector. Choose any $\delta \in \mathcal{SD}$. Let δ be a primitive normal vector of an $(n-1)$-dimensional space spanned by $d_1, \cdots, d_{n-1} \in \mathcal{D}$. We need to consider the convolution u of all u^j defined by (4) and the associated Minkowski sum D^{Ms} given by (6). It follows from Lemma 4 that D^{Ms} has the same demand type $\mathcal{D}$ as every bidder has.

Regarding $\mathcal{L}(p(t) + \varepsilon\delta)$ as a function in $\varepsilon \geq 0$, we have a function that changes linearly as ε increases from 0 up to the point $\varepsilon = \varepsilon^* > 0$ where $D^{Ms}(p(t) + \varepsilon\delta) \setminus D^{Ms}(p(t)) \neq \emptyset$. This is equivalent to that for all $j \in B$ we have $D^j(p(t) + \varepsilon\delta) \subseteq D^j(p(t))$ $(\forall \varepsilon \in [0, \varepsilon^*))$ and for some $j \in B$ we have $D^j(p(t) + \varepsilon^*\delta) \setminus D^j(p(t)) \neq \emptyset$. Also note that $D^{Ms}(p(t) + \varepsilon\delta)$ remains the same for all $\varepsilon \in (0, \varepsilon^*)$. Observe that if such a point ε^* does not exist, we consider $\varepsilon^* = +\infty$ and we can choose $\varepsilon = 1 < \varepsilon^*$ in the following argument. Hence we assume such a finite ε^* always exists. Then there exist some $d^* \in \mathcal{D}$ and an element (a vertex) x^* of $D^{Ms}(p(t)+\varepsilon\delta)$ for $0 < \varepsilon < \varepsilon^*$ such that $x^*+d^* \in D^{Ms}(p(t)+\varepsilon^*\delta)\setminus D^{Ms}(p(t))$ and for the convolution u of all u^j we have

$$u(x^* + d^*) = (p(t) + \varepsilon^*\delta) \cdot ((x^* + d^*) - x^*) + u(x^*). \tag{29}$$

From this we have

$$\varepsilon^*\delta \cdot d^* = u(x^* + d^*) - u(x^*) - p(t) \cdot d^*. \tag{30}$$

Moreover, we can see that d^* is not spanned by $d_1, \cdots, d_{n-1}$. Hence $d_1, \cdots, d_{n-1}, d^*$ is linearly independent and we have $\delta \cdot d^* > 0$ due to the definition of d^*. It follows from Lemma 5 that we have

$$0 < \delta \cdot d^* \leq 1. \tag{31}$$

Since the right-hand side of (30) is a non-zero integer, we see from (30) and (31) that $\varepsilon^* \geq 1$.

Since $\delta \in \mathcal{SD}$ is chosen arbitrarily in the above argument, we see that for each $\delta \in \mathcal{SD}$ the function $\mathcal{L}(p(t)+\varepsilon\delta)$ in ε is linear on the interval $[0, 1]$. Hence $\mathcal{L}(p(t)+\delta')$ as a function in δ' is a polyhedral conical convex function restricted on Conv($\mathcal{SD}$). This implies that equation (8) holds. □

Proof of Corollary 1: We see from the proof of Proposition 2 that $\mathcal{L}(p(t) + \delta')$ as a function in δ' is a polyhedral conical convex function restricted on Conv($\mathcal{SD}$) and is generated by function values $\mathcal{L}(p(t) + \varepsilon\delta)$ for all $\varepsilon \in [0, 1]$ and all $\delta \in \mathcal{SD}$. Hence the set of solutions to the left-side problem of (8) is a nonempty integral polytope. □

Proof of Corollary 2: The proof of Proposition 2 implies that $D^j(p+\varepsilon\delta) \subseteq D^j(p)$ and hence

$$x^j \in \arg\min_{x \in D^j(p)} x \cdot \delta$$

lies in $D^j(p + \varepsilon\delta)$ for all $\varepsilon \in [0, 1)$. If $D^j(p + \delta) \not\subseteq D^j(p)$, then we have

$$D^j(p) \cap D^j(p + \delta) = \arg\min_{x \in D^j(p)} \delta \cdot x = \arg\max_{x \in D^j(p+\delta)} \delta \cdot x. \tag{32}$$

Hence $x^j \in \arg\min_{x \in D^j(p)} x \cdot \delta$ lies in $D^j(p + \delta)$. □

Proof of Theorem 2: By Theorem 1 the auction market has a competitive equilibrium with an integral equilibrium price vector. Then by Proposition 1 of Ausubel (2006) and Lemma 1 of Sun and Yang (2009), a vector is a competitive equilibrium price vector if and only if it is a minimizer of the Lyapunov function.

Obviously, if p^* is a minimizer of the Lyapunov function $\mathcal{L}$, clearly it holds $\mathcal{L}(p^*) \leq \mathcal{L}(p^* + \delta)$ for all $\delta \in \mathcal{SD}$.

Assume now that $\mathcal{L}(p^*) \leq \mathcal{L}(p^* + \delta)$ for all $\delta \in \mathcal{SD}$. We claim that $\mathcal{L}(p) \geq \mathcal{L}(p^*)$ for all $p \in \mathbb{R}^N$. Then p^* is a minimizer of the Lyapunov function $\mathcal{L}$. Suppose to the contrary that there exists some $p \neq p^*$ such that $\mathcal{L}(p) < \mathcal{L}(p^*)$. Since $\text{Conv}(\mathcal{SD})$ is a full-dimensional convex set in $\mathbb{R}^N$ containing the n-vector $\mathbf{0}$ of zeros in its interior and so the set $\{p^*\} + \text{Conv}(\mathcal{SD})$ is also a full-dimensional convex set in $\mathbb{R}^N$ having p^* in its interior, one can easily take a strictly convex combination p' of p and p^* by choosing a sufficiently small $\alpha \in (0, 1)$ such that $p' = \alpha p + (1 - \alpha)p^* \in \{p^*\} + \text{Conv}(\mathcal{SD})$ and is close to p^*. Because of the convexity of $\mathcal{L}(\cdot)$, $\alpha > 0$, and $\mathcal{L}(p) - \mathcal{L}(p*) < 0$, we have

$$\mathcal{L}(p') \leq \alpha\mathcal{L}(p) + (1 - \alpha)\mathcal{L}(p^*) = \mathcal{L}(p^*) + \alpha(\mathcal{L}(p) - \mathcal{L}(p^*)) < \mathcal{L}(p^*). \tag{33}$$

It follows immediately from Proposition 2 and inequality (33) that

$$\min_{\delta \in \text{Conv}(\mathcal{SD})} \mathcal{L}(p^* + \delta) = \min_{\delta \in \mathcal{SD}} \mathcal{L}(p^* + \delta) \leq \mathcal{L}(p') < \mathcal{L}(p^*)$$

contradicting the hypothesis. This shows that $\mathcal{L}(p^*) \leq \mathcal{L}(p)$ holds for all $p \in \mathbb{R}^N$ and so p^* is a minimizer of the Lyapunov function $\mathcal{L}$, i.e., a competitive equilibrium price vector.

□

Proof of Corollary 3: By Theorem 1, the set of competitive equilibrium price vectors is a nonempty integral polytope. By assumption, p is not a minimizer of $\mathcal{L}$, i.e., p is not a competitive equilibrium price vector. Suppose to the contrary that there is no $\delta \in \mathcal{SD}$ such that $\mathcal{L}(p + \delta) < \mathcal{L}(p)$. Then we must have $\mathcal{L}(p) \leq \mathcal{L}(p + \delta)$ for all $\delta \in \mathcal{SD}$. By Theorem 2, p is a minimizer of the Lyapunov function $\mathcal{L}$, contradicting the assumption.

□

Proof of Theorem 3: Because the Lyapunov function $\mathcal{L}(\cdot)$ is convex and bounded from below and has a minimizer, any minimizer of the Lyapunov function is a competitive equilibrium price vector. Since the prices and value functions take only integer values and the UCD auction lowers the value of the Lyapunov function by a positive integer value in each round, the process must terminate in finite rounds, i.e., $\delta(t^*) = \mathbf{0}$ in Step 2 for some $t^* \in \mathbb{Z}_+$. Let $p(0), p(1), \cdots, p(t^*)$ be the generated finite sequence of price vectors. Clearly, we must have $\mathcal{L}(p(t^*)) \leq \mathcal{L}(p(t^*) + \delta)$ for all $\delta \in \mathcal{SD}$. Otherwise, we would have $\mathcal{L}(p(t^*)) > \mathcal{L}(p(t^*) + \delta)$ for some $\delta \in \mathcal{SD}$ with $\delta \neq \mathbf{0}$, contradicting $\delta(t^*) = \mathbf{0}$. It follows from Theorem 2 that $p(t^*)$ is a minimizer of the Lyapunov function, i.e., a competitive equilibrium price vector. □

Proof of Theorem 4: Because every bidder $j \in B$ bids straightforwardly according to his true UTD $\mathcal{D}$ function u^j and Assumptions (A1) and (A2) are satisfied, by Theorem 3 of Section 4 the auction finds a competitive equilibrium $(p^k(T^k), X^k)$ in every market $\mathcal{M}_{-k}$, $k \in B_0$. As bidders act truthfully, then for every bidder $j \in B_{-k}$ in every market $\mathcal{M}_{-k}$ at any time $t \in \mathbb{Z}_+$ we have $B^j_k(t) = D^j(p^k(t))$. It further follows from (12) in Section 4 that

$$\Delta^k_j(t) = \min_{x^j \in B^j_k(t)} x^j \cdot \delta^k(t) = V^j(p^k(t)) - V^j(p^k(t+1)).$$

By the rule in Step 3 of the auction, every bidder $j \in B$ pays β_j of (17) for the bundle $x^{0,j}$ assigned to him. It will be shown that β_j is actually equal to the VCG payment of bidder j given by $\beta^*_j = u^j(x^{j,0}) - R(N) + R_{-j}(N)$, where $R(N) = \sum_{h \in B} u^j(x^{0,h})$ and $R_{-j}(N) = \sum_{h \in B_{-j}} u^j(x^{j,h})$. Recall that $p^k(0) = p(0)$ for every $k \in B_0$. It follows from (17) that

$$\begin{aligned}
\beta_j &= \textstyle\sum_{h \in B_{-j}} \Big[\Big(\sum_{t=0}^{T^0-1} \Delta^0_h(t) - \sum_{t=0}^{T^j-1} \Delta^j_h(t)\Big) + x^{j,h} \cdot p^j(T^j) - x^{0,h} \cdot p^0(T^0)\Big] \\
&= \textstyle\sum_{h \in B_{-j}} \Big(\sum_{t=0}^{T^0-1} (V^h(p^0(t)) - V^h(p^0(t+1))) \\
&\qquad \textstyle - \sum_{t=0}^{T^j-1} (V^h(p^j(t)) - V^h(p^j(t+1)))\Big) \\
&\quad \textstyle + \sum_{h \in B_{-j}} x^{j,h} \cdot p^j(T^j) - \sum_{h \in B_{-j}} x^{0,h} \cdot p^0(T^0) \\
&= \textstyle\sum_{h \in B_{-j}} \Big((V^h(p^0(0)) - V^h(p^0(T^0))) - (V^h(p^j(0)) - V^h(p^j(T^j)))\Big) \\
&\qquad \textstyle + \sum_{h \in B_{-j}} x^{j,h} \cdot p^j(T^j) - \sum_{h \in B_{-j}} x^{0,h} \cdot p^0(T^0) \\
&= \textstyle\sum_{h \in B_{-j}} \Big(V^h(p^j(T^j)) + x^{j,h} \cdot p^j(T^j)\Big) - \sum_{h \in B_{-j}} \Big(V^h(p^0(T^0)) + x^{0,h} \cdot p^0(T^0)\Big) \\
&= \textstyle\sum_{h \in B_{-j}} u^j(x^{j,h}) - \sum_{h \in B_{-j}} u^j(x^{0,h}) = u^j(x^{0,j}) - R(N) + R_{-j}(N) = \beta^*_j.
\end{aligned}$$

□

Proof of Theorem 5: Consider any time $\hat{t} \in \mathbb{Z}_+$, any history profile $\{H^{\hat{t}}_h\}_{h \in B}$, and any realization $\{u^h\}_{h \in B}$ of profile of utility functions of private information. Clearly, the outcome of the game depends on the histories $H^{\hat{t}}_h$ for $h \in B$ and actions that bidders will take in the continuation game starting from $\hat{t}$. Note that bidders cannot change histories but can influence the path of the future from $\hat{t}$ on. Take any player $j \in B$. Suppose that in the continuation game from time $\hat{t}$ on, every opponent $h \in B_{-j}$ of player j bids sincerely at any $t \in \mathbb{Z}_+(t \geq \hat{t})$ and in every market $\mathcal{M}_{-k}$ for $k \in B_0$, namely,

$$\sigma_h(t, k, H_h^t) = B_k^h(t) = D^h(p^k(t)) = \arg \max_{x \in \{0,1\}^N} \{u^h(x) - x \cdot p^k(t)\}.$$

It implies that for every bidder $h \in B_{-j}$ in the markets $\mathcal{M}_{-j}$ and $\mathcal{M}$ at every time $t \geq \hat{t}$

$$\Delta_h^j(t) = \min_{x^h \in B_j^h(t)} x^h \cdot \delta^h(t) = V^h(p^j(t)) - V^h(p^j(t+1))$$

and $\Delta_h^0(t) = \min_{x^h \in B_0^h(t)} x^h \cdot \delta^h(t) = V^h(p^0(t)) - V^h(p^0(t+1))$. However, the above equations do not necessarily hold true for time $t < \hat{t}$.

Clearly, in this continuation game from time $\hat{t}$, when all opponents of player j choose sincere bidding strategies, because of the option of walking away in Step 3, bidder j prefers a strategy which causes the auction to stop at Step 3 and yields a nonnegative payoff to him, to any other strategy which leads the auction to the broken down case and gives him a strictly negative payoff of $-c < 0$. Therefore, it is sufficient to compare the sincere bidding strategy with any other strategy which leads the auction to Step 3. Suppose that $\sigma_j'(\cdot, \cdot, \cdot \mid \hat{t}, k, H_j^{\hat{t}})$ (σ_j' in short) for all $k \in B_0 \setminus \{j\}$ is such a continuation strategy of player j resulting in an allocation $(y^{0,h}, h \in B)$ in the market $\mathcal{M}$, and that bidder j's (continuation) sincere bidding strategy results in an allocation $(x^{0,h}, h \in B)$ in the market $\mathcal{M}$ by Theorem 3 in Section 4. Without any loss of generality, we assume that by the time $\hat{t}$, the auction has not found any allocation in the markets $\mathcal{M}$ and $\mathcal{M}_{-j}$, i.e., $\hat{t} < T^{-0}$ and $\hat{t} < T^{-j}$. When player j chooses the strategy σ_j', his payment β_j' given by (17) is

$$\begin{aligned}
\beta_j' &= \textstyle\sum_{h \in B_{-j}} \Big[\Big(\sum_{t=0}^{T^0-1} \Delta_h^0(t) - \sum_{t=0}^{T^j-1} \Delta_h^j(t) \Big) + x^{j,h} \cdot p^j(T^j) - y^{0,h} \cdot p^0(T^0) \Big] \\
&= \textstyle\sum_{h \in B_{-j}} \Big(\sum_{t=0}^{\hat{t}-1} \Delta_h^0(t) + \sum_{t=\hat{t}}^{T^0-1} \Delta_h^0(t) - \sum_{t=0}^{\hat{t}-1} \Delta_h^j(t) - \sum_{t=\hat{t}}^{T^j-1} \Delta_h^j(t) \Big) \\
&\quad \textstyle + \sum_{h \in B_{-j}} x^{j,h} \cdot p^j(T^j) - \sum_{h \in B_{-j}} y^{0,h} \cdot p^{0,h}(T^0) \\
&= \textstyle\sum_{h \in B_{-j}} \Big[\sum_{t=0}^{\hat{t}-1} \Delta_h^0(t) + \sum_{t=\hat{t}}^{T^0-1} (V^h(p^0(t)) - V^h(p^0(t+1))) \\
&\qquad \textstyle - \sum_{t=0}^{\hat{t}-1} \Delta_h^j(t) - \sum_{t=\hat{t}}^{T^j-1} (V^h(p^j(t)) - V^h(p^j(t+1))) \Big] \\
&\quad \textstyle + \sum_{h \in B_{-j}} x^{j,h} \cdot p^j(T^j) - \sum_{h \in B_{-j}} y^{0,h} \cdot p^{0,h}(T^0) \\
&= \textstyle\sum_{h \in B_{-j}} \Big(\sum_{t=0}^{\hat{t}-1} [\Delta_h^0(t) - \Delta_h^j(t)] + V^h(p^0(\hat{t})) + V^h(p^j(T^j)) - V^h(p^j(\hat{t})) \Big) \\
&\qquad \textstyle + \sum_{h \in B_{-j}} x^{j,h} \cdot p^j(T^j) - \Big(\sum_{h \in B_{-j}} V^h(p^0(T^0)) + \sum_{h \in B_{-j}} y^{0,h} \cdot p^0(T^0) \Big) \\
&= \textstyle\Gamma_{-j} - \sum_{h \in B_{-j}} u^h(y^{0,h}),
\end{aligned}$$

where Γ_{-j} is given by

$$\begin{aligned}
\Gamma_{-j} &= \textstyle\sum_{h \in B_{-j}} \Big[\sum_{t=0}^{\hat{t}-1} \Big(\Delta_h^0(t) - \Delta_h^j(t) \Big) + V^h(p^0(\hat{t})) + V^h(p^j(T^j)) - V^h(p^j(\hat{t})) \\
&\quad + x^{j,h} \cdot p^j(T^j) \Big].
\end{aligned}$$

Observe that Γ_{-j} is totally determined by the history profile $\{H_h^{\hat{t}}\}_{h\in B}$ and the market $\mathcal{M}_{-j}$ without bidder j, and does not depend on player j's strategy σ'_j. Similarly, we can prove that if bidder j adopts the sincere bidding strategy, his payment $\hat{\beta}_j$ will be

$$\hat{\beta}_j = \Gamma_{-j} - \sum_{h\in B_{-j}} u^h(x^{0,h}).$$

Moreover it follows from Theorem 3 in Section 4 that when bidders bid truthfully according to their utility functions u^h, $h \in B$, and Assumptions (A1) and (A2) are satisfied, the allocation $(x^{0,h}, h \in B)$ in the market $\mathcal{M}$ found by the auction will be efficient. That is,

$$u^j(x^{0,j}) + \sum_{h\in B_{-j}} u^h(x^{0,h}) \geq u^j(y^{0,j}) + \sum_{h\in B_{-j}} u^h(y^{0,h}).$$

Taking the option of walking away into every bidder's account together with the above discussion gives the payoff $\hat{\mathcal{P}}_j$ of bidder j in the case of using the sincere bidding strategy and his payoff $\mathcal{P}'_j$ in the case of using the strategy σ'_i as follows

$$\begin{aligned}
\hat{\mathcal{P}}_j &= \max\{u^j(x^{0,j}) - \hat{\beta}_j, 0\} \\
&= \max\{u^j(x^{0,j}) - (\Gamma_{-j} - \textstyle\sum_{h\in B_{-j}} u^h(x^{0,h})), 0\} \\
&= \max\{u^j(x^{0,j}) + \textstyle\sum_{h\in B_{-j}} u^h(x^{0,h}) - \Gamma_{-j}, 0\} \\
&\geq \max\{u^j(y^{0,j}) + \textstyle\sum_{h\in B_{-j}} u^h(y^{0,h}) - \Gamma_{-j}, 0\} \\
&= \max\{u^j(y^{0,j}) - \beta'_j, 0\} = \mathcal{P}'_j.
\end{aligned}$$

This demonstrates that every player's sincere bidding strategy is indeed his ex post perfect strategy. Therefore bidding sincerely by every bidder is an ex post perfect equilibrium.

□

Proof of Proposition 4: It follows from the proof of Theorem 4 that every bidder $j \in B$ receives bundle $x^{0,j}$ and pays β_j^* and his net profit equals

$$\begin{aligned}
u^j(x^{0,j}) - \beta_j^* &= R(N) - R_{-j}(N) = \textstyle\sum_{h\in B_0} u^h(x^{0,h}) - \sum_{h\in B_{-j}} u^h(x^{j,h}) \\
&= \textstyle\sum_{h\in B_0} u^h(x^{0,h}) - \sum_{h\in B_0} u^h(x^{j,h}) \geq 0
\end{aligned}$$

where $x^{j,j} = 0$.

We now prove that the auction is also beneficial to the seller. First, consider the case that u^0 is submodular. Recall that for every $k \in B$, $(x^{k,h}, h \in B_{-k})$ is the equilibrium allocation in market $\mathcal{M}_{-k}$ found by the auction. By definition, it is easy to see that

$$R_{-j}(N) = \sum_{h\in B_{-j}} u^h(x^{j,h}) \geq \sum_{h\in B\setminus\{j\}} u^h(x^{0,h}) + u^0(x^{0,0} + x^{0,j}).$$

The utility $\tilde{\mathcal{P}}_0$ received by the seller equals

$$\begin{aligned}
\tilde{\mathcal{P}}_0 &= u^0(x^{0,0}) + \textstyle\sum_{j\in B}\beta_j^* \\
&= \textstyle\sum_{j\in B}\Big(u^j(x^{0,j}) - R(N) + R_{-j}(N)\Big) = \sum_{j\in B} R_{-j}(N) - (m-1)R(N) \\
&\geq \textstyle\sum_{j\in B}\Big(u^0(x^{0,0}+x^{0,j}) + \sum_{h\in B\setminus\{j\}} u^h(x^{0,h})\Big) - (m-1)R(N) \\
&= \textstyle\sum_{j\in B} u^0(x^{0,0}+x^{0,j}) - (m-1)u^0(x^{0,0}).
\end{aligned}$$

Then submodularity implies that for every $j = 1, 2, \cdots, m-1$ we have

$$u^0(\sum_{h=0}^{j} x^{0,h}) + u^0(x^{0,0}+x^{0,j+1}) \geq u^0(\sum_{h=0}^{j+1} x^{0,h}) + u^0(x^{0,0}).$$

Summing up these inequalities leads to

$$\sum_{j\in B} u^0(x^{0,0}+x^{0,j}) \geq u^0(\sum_{j\in B_0} x^{0,j}) + (m-1)u^0(x^{0,0})$$

from which we have

$$\tilde{\mathcal{P}}_0 = \sum_{j\in B} u^0(x^{0,0}+x^{0,j}) - (m-1)u^0(x^{0,0}) \geq u^0(\sum_{j\in B_0} x^{0,j}) = u^0(N).$$

So the utility the seller receives from trading is at least as good as she does not trade.

Second, consider the case that u^0 is superadditive. For every $j \in B$ we have

$$R_{-j}(N) = \sum_{h\in B_{-j}} u^h(x^{j,h}) \geq \sum_{h\in B\setminus\{j\}} u^h(x^{0,h}) + u^0(x^{0,0}+x^{0,j})$$

and $u^0(x^{0,0}+x^{0,j}) \geq u^0(x^{0,0}) + u^0(x^{0,j})$. Then the utility $\tilde{\mathcal{P}}_0$ received by the seller equals

$$\begin{aligned}
\tilde{\mathcal{P}}_0 &= u^0(x^{0,0}) + \textstyle\sum_{j\in B}\beta_j^* \\
&= u^0(x^{0,0}) + \textstyle\sum_{j\in B}[u^j(x^{0,j}) - R(N) + R_{-j}(N)] \\
&= u^0(x^{0,0}) + \textstyle\sum_{j\in B}\Big(u^j(x^{0,j}) - (u^0(x^{0,0}) + \sum_{h\in B} u^h(x^{0,h})) + R_{-j}(N)\Big) \\
&= u^0(x^{0,0}) + \textstyle\sum_{j\in B}[R_{-j}(N) - (u^0(x^{0,0}) + \sum_{h\in B\setminus\{j\}} u^h(x^{0,h}))] \\
&= u^0(x^{0,0}) + \textstyle\sum_{j\in B}\Big(u^0(x^{0,j}) + R_{-j}(N) - (u^0(x^{0,0}) + u^0(x^{0,j}) + \sum_{h\in B\setminus\{j\}} u^h(x^{0,h}))\Big) \\
&\geq u^0(x^{0,0}) + \textstyle\sum_{j\in B}[u^0(x^{0,j}) + R_{-j}(N) - (u^0(x^{0,0}+x^{0,j}) + \sum_{h\in B\setminus\{j\}} u^h(x^{0,h}))] \\
&\geq u^0(x^{0,0}) + \textstyle\sum_{j\in B}\Big(u^0(x^{0,j}) + R_{-j}(N) - R_{-j}(N)\Big) = \sum_{j\in B_0} u^0(x^{0,j}).
\end{aligned}$$

This shows that the payment β_j^* received by the seller for every sold bundle $x^{0,j}$ is at least as big as its reserve price $u^0(x^{0,j})$. We are done. □

Proof of Proposition 5: Because every bidder has the option of walking away in Step 3 and faces no punishment in Step 4, his final payoff cannot be negative if he is able to judge between positive and negative numbers, not necessarily acting optimally. Consequently, the ICUD auction is ex post individually rational. □

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