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How sensitive are the results in voting theory when just one other voter joins in? Some instances with spatial majority voting*

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Abstract

In this paper we consider situations of (multidimensional) spatial majority voting. We explore some possibilities such that under some regularity assumptions usual in this literature, if the number of voters changes from being odd to even then some results may change somewhat drastically. For example, we show that with an even number of voters if the core of the voting situation is singleton (and the core element is in the interior of the policy space) then the core is never externally stable (i.e., the situation has no Condorcet winner). This is sharply opposite to what happens with an odd number of voters: in that case, under identical assumptions on the primitives, it is well known that if the core of the voting situation is non-empty then the singleton core is always externally stable: i.e., the core element is the Condorcet winner majority-dominating every other policy vector. We find similar strikingly contrasting results with respect to the coincidence of the core and the (Gillies) uncovered set and the size and geometry of the (Gillies) uncovered set. These results rectify some erroneous statements found in this literature.

Keywords: Spatial Voting Situations; Core; Condorcet winner; Uncovered set.

JEL Classification: D71; C71.

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1 Introduction

It is quite well-known that for a voting situation if the number of voters changes from being odd to even then, *everything else remaining the same*, the properties associated with the voting situation may change. Perhaps the simplest and the most well-known example of such "may change" occurrences is the following. Take a finite set of voters and a finite set of candidates and suppose that each voter has a strict preference ordering over the set of candidates. Then, if there is an odd number of voters then, with majority rule voting, for any two candidates x and y, either x is socially preferred to y or the converse is true: i.e., the resulting social ordering is decisive between any two candidates. But that may not be the case if there is an even number of voters.

In this paper we provide some somewhat drastic examples in such spirit in the context of multi-dimensional *spatial* voting with majority rule. Under some regularity assumptions usual in this literature we show that with an even number of voters if the core of the voting situation is singleton (and if the core element is in the interior of the policy space) then the core is never externally stable (i.e., in other words, a Condorcet winner for this situation does not exist). This is sharply opposite to what happens with an odd number of voters: in that case, *under identical assumptions on the primitives*, it is well known that if the core of the voting situation is non-empty then the singleton core is always externally stable: i.e., the core element majority-dominates every other policy vector. We find similar strikingly contrasting results with respect to the coincidence of the core and the (Gillies) uncovered set and the size and geometry of the (Gillies) uncovered set.

Here we point out a few notable features of our exercise.

At first sight, indeed, one might think that when one item of the primitive components of an environment changes then it is unsurprising that results may change as well. In response we would like to point out that our results do not merely illustrate *one/some "may change" case(s).* We show that if the number of voters changes from being odd to even then, under some usual regularity assumptions, *for non-negligible sets of the possible cases* some properties get, somewhat "dramatically", *exactly opposite.*

Next, given our initial regularity assumptions (detailed in the next section) it is known that if a voting situation, with an odd number of voters, has a non-empty core then the core is singleton. Since our focus in this work is to check for sensitivity of some results when the number of voters changes from being odd to even, our results are for voting situations having a singleton core. Given this, one central message of this work is as follows. A singleton core gives a sharp and intuitive prediction for a voting situation: no coalition is likely to upset this unique policy-vector in place. With an odd number of voters this predicted policy-vector possesses some further desirable properties. But these properties get lost when the number of voters becomes even.

Finally, we find that with respect to our results, even in quite well-known works authors made erroneous statements. Instead of making a compilation of such examples of incorrect statements here in this introductory section itself, we indicate those, for clarity, in remarks following the corresponding results of ours.

The next section gives the primitives of our framework and some preliminary pieces of notation. The substantive results and discussions around those are given in Sections 3, 4 and 5. Section 6 provides a couple of concluding remarks. An appendix as the penultimate section provides the detailed proofs of the results. At the very end are some figures which, we hope, might help in visualizing the arguments behind the proofs of some of the results.[1]

2 The primitives and some preliminary notation

Let $Z \subseteq \mathbb{R}^k$ be a compact full-dimensional convex subset of some finite (k-)dimensional Euclidean space with $k \geq 1$ (in what follows, the underlying topological space is taken to be the entire $\mathbb{R}^k$). This set, Z, is identified to be the feasible set of (possibly multi-dimensional) policies on which a voter votes. In what follows, because we have to use geometrical arguments quite a bit, we shall often call a policy simply a "point".

Let N be the finite set of players or voters. For a set A, by $|A|$ we would denote its cardinality. For each $i \in N$ the preferences of i on Z is represented by a real-valued continuously differentiable and strictly concave pay-off function $u_i : Z \mapsto \mathbb{R}$. The spatial voting situation we consider below is obtained by introducing the method of majority rule voting.

Definition 2.1 (Domination by Majority Rule) *Given $x, y \in Z$, the policy x dominates policy y via coalition $S \subseteq N$, if and only if $|S| > |N|/2$ and $u_i(x) > u_i(y)$ for each $i \in S$. We denote this as $x \succ_S y$. If there exists a majority coalition S via which x dominates y, we denote that as $x \succ y$.*

The collection $G = \langle Z, N, (u_i)_{i \in N} \rangle$ is a *spatial voting situation with majority rule* (which we shall often refer below simply as a *voting situation* with no possibility of confusion).

[1]Please note that these figures are for illustrative purposes only–these have not been, at least as yet, constructed using or used for any rigorous measurements.

For any $x \in Z$ and $i \in N$, by $C_i(x)$ we denote the set $\{y \in Z \,:\, u_i(y) \geq u_i(x)\}$, i.e., the upper contour set of i for x. Similarly, for any $S \subseteq N$ and any $x \in Z$, by $C_S(x)$ we denote the set $\{y \in Z \,:\, u_i(y) \geq u_i(x)$ for each $i \in S\}$. In Section 5 below we shall have to deal a lot with coalitions containing only 2 voters for which a special piece of notation may be convenient. For a coalition of 2 voters (say, $\{i, j\}$) and any $x \in Z$, by $C_{ij}(x)$ we denote the set $\{y \in Z \,:\, u_m(y) \geq u_m(x)$ for each $m \in \{i, j\}\}$. Also, for any $x \in Z$ and $i \in N$, by $I_i(x)$ we denote the set $\{y \in Z \,:\, u_i(y) = u_i(x)\}$.

For any set $A \subseteq Z$, by $cl(A)$, $int(A)$ and $bd(A)$ we denote the closure of A, the interior of A and the boundary of A respectively. Also, for any two points $x, y \in Z$, by $\rho(x, y)$ we denote the (Euclidean) distance between these two points.

For $i \in N$, we denote the unique maximizer of u_i on Z, the *ideal point* of i in Z, by $\bar{x}_i$.

3 On external stability of a singleton core

We start by recalling some of the relevant definitions.

Definition 3.1 (The core of a voting situation) *The core of a voting situation G is the subset $K = \{y \in Z \,:\, \nexists x \in Z$ such that $x \succ y\}$.*

Definition 3.2 (External stability) *Given a voting situation G, a set $V \subseteq Z$ is said to be* externally stable *if for every $x \in Z \setminus V$ there exists $y \in V$ such that $y \succ x$.*

Recall that a point $x \in Z$ is said to be the *Condorcet winner* of the voting situation G, if for any other policy $y \neq x$, $x \succ y$. Recall that if a voting situation admits a Condorcet winner, then it is the unique element in the core of that situation.

Our main result in this section is:

Proposition 1 *Consider a voting situation G for which $|N|$ is an even positive integer. Suppose further that for G, the core $K = \{x_0\}$ is singleton and the point x_0 is in the interior of Z.*
(i) Assume additionally that for at most one $i \in N$ is it the case that $x_0 = \bar{x}_i$. Then x_0 cannot be the Condorcet winner of G.
(ii) Assume, rather, that for at least two distinct $i, j \in N$ is it the case that $x_0 = \bar{x}_i = \bar{x}_j$. Then on one hand there exists a voting situation G_1 for which x_0 is the Condorcet winner of G_1; on the other hand there exists another voting situation

G_2 for which x_0 is not the Condorcet winner of G_2.

Remark 3.1 Note that if $|N| = 2$, i.e., there are only two voters, (i, j), for the situation then the core is singleton if and only if the element in the core $x_0 = \bar{x}_i = \bar{x}_j$.

Recall the respective property of the core under identical primitives when $|N|$ is odd:

Proposition $1'$ *Consider a voting situation G for which $|N|$ is an odd positive integer. Suppose further that for G the core K is non-empty. Then there is a unique element in the core, x_0, which is the Condorcet winner of G.*

A proof of Proposition $1'$ is given in Cox (1987, p. 411).

Recall that all the detailed proofs, including that of Proposition 1, are collected in the Appendix (and at the very end, in the Section entitled "Figures" we provide three diagrams–Figures 1 to 3–illustrating some arguments for proving this Proposition, which might be helpful).

We conclude this section with a couple of remarks on Proposition 1.

Remark 3.2 Note that the condition in part (i) of Proposition 1 that x_0, the unique element in the core, is in the interior of Z, and that for *at most* one $i \in N$ is it the case that $x_0 = \bar{x}_i$ is *not* pathological as Z is a convex subset of $\mathbb{R}^k$ of full dimension.

Remark 3.3 With respect to Proposition 1 one example of erroneous writing is as follows. Discussing a framework which is more general than ours Penn (2009) writes: "...the uncovered set, minimal covering set, tournament equilibrium set, Banks set, largest consistent set, and von Neumann-Morgenstern stable set, as all of these sets reduce to the core, if one exists...". Proposition 1 demonstrates that the statement is wrong with respect to von Neumann-Morgenstern stable sets.

4 (Non-)coincidence of the core and the (Gillies) uncovered set

Recall that the idea of several kinds of "covering" and the corresponding "uncovered" sets as predictions for voting situations became popular from onwards the 1970s (Duggan, 2013 provides a comprehensive review of these ideas). One appealing such prediction is the (Gillies) uncovered set (which below we shall often refer simply as "the uncovered set" with no possibility of confusion).

Definition 4.1 (The (Gillies) uncovered set) *Let $x, y \in Z$. We say that x covers y, denoted as $y \prec_c x$ if the following hold:*

$$x \succ y;$$
$$z \in Z, \;\; z \succ x \Longrightarrow z \succ y.$$

The uncovered set is given by $UC = \{y \in Z \,:\, \nexists z$ such that $y \prec_c z\}$.

UC has some nice properties. For example, even under conditions weaker than the primitives of this work it is always non-empty (Bordes et al., 1992). Further, the uncovered set is also *Condorcet-consistent*: in the sense that if a majority voting situation G has a Condorcet winner then the UC coincides with that Condorcet winner.

As a straightforward Corollary of Proposition 1 we obtain the following result (the proof of which, as usually, is given in the Appendix).

Corollary 1 *For any voting situation G obeying the Conditions of part (i) of Proposition 1, $K \neq UC$.*

This is in sharp contrast to what happens when $|N|$ is odd:

Proposition 1″ (Cox, 1987) *Consider a voting situation G with $|N|$ odd. Then, if $K \neq \emptyset$, then $K = UC$.*

Remark 4.1 As we did in Remark 3.3 above, here we give a couple of examples of erroneous writing with respect to Corollary 1 too. In a framework similar to ours, with respect to the (Gillies) uncovered set Cox (1987) writes: "...the uncovered set collapses to the core, when one exists". Similarly, in their well-known two-volume textbook Austen-Smith and Banks write (p. 274 of vol. 2; 2005): "...the uncovered set coincides with the core when the latter is nonempty and singleton" (the definition of the uncovered set they use, in fact, gives a superset of the uncovered set with which we have worked here). These statements are untenable in view of Corollary 1.

Remark 4.2 As one illustration of the messages of part (i) of Proposition 1 and Corollary 1 consider the following voting situation G'.
$N = \{1, 2, 3, 4, 5\}$. The set of policies, $Z = \{x \in \mathbb{R}^2 | x_1 \in [-1, 1]; x_2 \in [-1, 1]\}$. Each player i has an ideal point $\bar{x}_i$ whose coordinates are given as follows. The point $\bar{x}_1 = (-1, -1)$; $\bar{x}_2 = (1, -1)$; $\bar{x}_3 = (1, 1)$, $\bar{x}_4 = (-1, 1)$ and $\bar{x}_5 = (0, 0)$. The voters' preferences are Euclidean, i.e., for any $i \in N$, and $x \in Z$, $u_i(x) = -(\rho(x, \bar{x}_i))^2$. It is easy to see that the core of G' is the singleton set containing the point $(0, 0)$. This core is externally stable and is the uncovered set as well.

Now drop the voter 5 and take the voting situation G such that *everything else remains as in the voting situation G' above.* Then, the core of G is still the singleton set containing the point $(0, 0)$. But this core is not externally stable (see the proof of Proposition 3.2 in Bhattacharya et al., 2018). Further, somewhat dramatically, *every* policy not dominated by the single element in the core belongs to the uncovered set of G (see Result 3.1 in Bhattacharya et al., 2018).
In the following section we explore this last feature in some more general detail.

5 How drastic can the non-coincidence be? On the size and geometry of the (Gillies) uncovered set

In continuation of the discussion in the previous section, note that if a policy x is dominated by a policy y that belongs to the core then x is covered by y as well. Thus, perhaps the maximally contrasting scenario could be a voting situation with an even number of voters for which every policy not dominated by some element in the core (and the elements in the core itself) belong to the uncovered set. In Remark 4.2 above we gave an example of a voting situation (with 4 voters) which has this feature. In this section we explore how far that specific example can be generalized: i.e., under what conditions every policy not dominated by the element in a singleton core is in the uncovered set.

Toward that goal we mainly consider a voting situation G which obeys following assumptions (A1 to A5 below) in addition to or as special cases of the items of primitive we assumed in Section 2 above and which has a singleton core containing a single point x_0 as well.

A1. The set of policy vectors, Z, is a compact convex subset of $\mathbb{R}^2$ and for every voter $i \in N$, its preferences on Z is Euclidean with a distinct ideal point (i.e., recall from Remark 4.2 above, for each $i \in N$ there exists a *distinct* $\bar{x}_i \in Z$, the ideal point of i in Z, such that for any policy $x \in Z$, $u_i(x) = -(\rho(x, \bar{x}_i))^2$).

In what follows, for any $i \in N$ and $y \in Z$, by $I_i(y)$ we denote the *indifference curve* of voter i through the point y: i.e., the set $\{z \in Z : u_i(z) = u_i(y)\}$. Recall that given A1, for each $i \in N$ and $y \in Z$, the curve $I_i(y)$ is a (circular) arc $\widehat{\bar{x}_i y}$ centred at $\bar{x}_i$ and passing through y.[2]

A2. The cardinality of N is $4q$ where q is a positive integer. Further, for every

[2]In this Proposition we will have to use circular arcs quite a lot. Given points $a, b \in Z$, by $\widehat{ab}$ we denote the circular arc centred at point a and passing through point b.

voter $i \in N$, $\bar{x}_i$, the ideal point of i in Z, is on the boundary of Z and no three such ideal points are collinear.

A3. The unique core point, x_0, is in the interior of Z (and thus, x_0 does not coincide with the ideal point of any of the voters).

Given A3 above, by, e.g., Duggan (2018; p.2) there is a set $\Omega \subset (N \times N)$ of $|N|/2$ pairs of distinct voters such that for every pair $(i, j) \in \Omega$ the normalized gradient of i's pay-off function at x_0 is the negative of the normalized gradient of j's pay-off function at x_0. For any two distinct voters m, p denote by K_{mp} the "mp-core", defined as follows: $K_{mp} = \{x \in Z$: there does not exist $y \in Z$ such that $u_m(y) > u_m(x)$ as well as $u_p(y) > u_p(x)\}$.
Given the assumption A1 above, it is easy to see that for any pair of distinct voters (m, p), K_{mp} is the straight line segment connecting $\bar{x}_m$ and $\bar{x}_p$. Further, it is also easy to see that given our assumptions above, the core point x_0 lies at the intersection of the straight line segments K_{mp}'s where each pair of voters (m, p) is in Ω such that for every voter m, its pairing voter p (for which $(m, p) \in \Omega$) is unique. We denote, for each such pair $(m, p) \in \Omega$, by $\omega(m)$ the voter p who is paired with m in Ω.

Next we assume:

A4. For each pair $(i, j) \in \Omega$, there exists a pair $(m, p) \in \Omega$ such that the line segments K_{ij} and K_{mp} are perpendicular to each other at x_0.

Further we assume:

A5. For any two distinct voters $m, p \in N$, the (circular) arcs $\overset{\frown}{\bar{x}_m x_0}$ and $\overset{\frown}{\bar{x}_p x_0}$ (centred, respectively, at $\bar{x}_m$ and $\bar{x}_p$) intersect within Z only at the single point x_0.

Note that the 4-voters example mentioned above–in Remark 4.2–satisfies each of A1 to A5.

Denote by $M(x_0)$ the set of points not dominated by x_0: i.e., $M(x_0) = \{y \in Z$: x_0 does not dominate $y\}$. Note that $M(x_0) = \cup_{S \in H} C_S(x_0)$ where H is the set of all coalitions containing $|N|/2$ voters. By P denote the subset of the coalition S's in H such that for every coalition S in P, $C_S(x_0) \neq \emptyset$ and $C_S(x_0) \neq \{x_0\}$.

Before getting into the main Proposition of this section we establish a preliminary necessary Lemma.

Lemma 2.0 *Consider a voting situation G which satisfies A1 to A5 given above. Take any coalition S in P (i.e., the $|N|/2$ voter coalition S is such that $C_S(x_0) \neq \emptyset$*

and $C_S(x_0) \neq \{x_0\}$). Then there exists a pair of distinct voters $i, t \in S$ such that $C_S(x_0) = C_{it}(x_0)$.

Now we state our final assumption.

A6. Take any $|N|/2$-voter coalition $T \in P$ and let $m, p \in T$ be the corresponding pair of voters such that $C_T(x_0) = C_{mp}(x_0)$ (Lemma 2.0 above ensures the existence of such a pair). Then K_{mp}, the mp-core straight line segment, is a subset of the boundary of Z.

Note that the example in Remark 4.2 obeys A6 as well.

We recognize that some of the assumptions among A1-A6 are quite stringent. However, this feature may not be entirely unappealing in our context of this Section because our goal is to identify some conditions under which the difference between the core and the uncovered set is maximal.

We obtain the following result.

Proposition 2 *(i) Consider a voting situation G which satisfies A1 to A5 given above. Then there exists a neighbourhood B of x_0 such that $B \cap M(x_0)$ is a subset of the Gillies uncovered set UC.*
(ii) If G additionally satisfies A6 then $M(x_0) = UC$.

We prove this Proposition through the six steps outlined below. The proof is constructive and a main driver behind the proof is Proposition 37 in Duggan (2013) which says, in words, that the (Gillies) uncovered set is the union of undominated subsets (with respect to the domination relation $\succ$) within the externally stable subsets of Z. While the detailed proofs of all these steps are collected, as before, in the Appendix, here we provide an overview of these steps.

Fix an arbitrary $|N|/2$-voter coalition $S \in P$ and recall from Lemma 2.0 above that $C_S(x_0) = C_{it}(x_0)$ for some pair of distinct voters $i, t \in S$.

Step 1: The set $C_{N \setminus S}(x_0) = C_{jw}(x_0)$ where $j = \omega(i)$ and $w = \omega(t)$.

Step 2: For voter $i \in S$, (resp. voter $t \in S$) $\bar{x}_i \notin C_t(x_0)$ (resp. $\bar{x}_t \notin C_i(x_0)$).

Next we define some subsets of Z which are crucial for our proof. Especially by Step 2 above, these subsets are well-defined. Figure 4 might be useful for visualizing these subsets.

Denote by I_i^+ (resp. I_t^+) the sub-arc of voter i's (resp. voter t's) indifference curve through x_0 which form the boundary of $C_S(x_0)$. The complement sub-arcs, $I_i(x_0) \setminus I_i^+$ and $I_t(x_0) \setminus I_t^+$, are denoted by I_i^- and I_t^- respectively. Similarly, denote by I_j^+ (resp. I_w^+) the sub-arc of voter j's (resp. voter w's) indifference curve through x_0 which form the boundary of $C_{N\setminus S}(x_0)$. The complement sub-arcs are denoted by I_j^- and I_w^- respectively.

Since $C_S(x_0) = C_{it}(x_0)$, K_{it}, the straight line segment connecting $\bar{x}_i$ and $\bar{x}_t$, does not pass through x_0. Denote by A_S the subset of $C_S(x_0)$ bounded by I_i^+, I_t^+ and K_{it}. By p_S denote the perpendicular from x_0 on K_{it} (by elementary geometry/trigonometry $p_S \subset A_S$). By A_i (resp. A_t) denote the subset of A_S bounded by I_i^+ (resp. I_t^+), p_S and K_{it}.

Now take any $y \in A_i$ (resp. A_t). Let $\alpha_y \subseteq A_i$ (resp. A_t) be the arc centred at $\bar{x}_i$ (resp. $\bar{x}_t$) passing through y, starting at T^y, the point at which this arc intersects p_S and ending at a point B^y on K_{it} (by A5 above this is assured; only when y is the point at which p_S intersects K_{it}, is α_y just a single point). Please note that for some point y, T^y may be x_0 itself. Denote by $p_y^T \subseteq p_S$ the straight line segment connecting x_0 and T^y. Further, denote by p_y^B the straight line segment starting at B^y, perpendicular to K_{it} and ending at the boundary of Z. Note that if the voting situation obeys A6 then $p_y^B = \{B_y\}$.

Construct the curve l_y passing through y as $p_y^T \cup \alpha_y \cup p_y^B$. Note that l_y can be parametrized by a continuous (invertible) function $\lambda_y : [0,1] \mapsto l_y$ such that $\lambda_y(0) = x_0$ and $\lambda_y(1)$ is the point at the boundary of Z at which l_y intersects the boundary of Z.

Step 3: Take any $y \in A_S$. Let $E_y \subset Z$ be $I_i^- \cup I_t^- \cup I_j^+ \cup I_w^+ \cup l_y$ where l_y has been specified in the previous paragraph (Figure 5 contains one illustration of the set E_y). Then E_y is externally stable.

Step 4: For every $y \in C_S(x_0)$ there exists no $z \in I_i^- \cup I_t^- \cup I_j^+ \cup I_w^+$ such that $z \succ y$.

Step 5: Take any $y \in A_S$. With respect to this y, let E_y be as specified in Step 3 above. Consider $l_y \subset E_y$. Then for no point x in $p_y^T \cup \alpha_y$ (call this subset γ_y) is it the case that some other point z in l_y, dominates x (i.e., the points in γ_y are undominated within l_y).

Step 6: Building on the Steps above, finally, by using Proposition 37 in Duggan (2013), we show that for each $S \in P$, $A_S \subseteq UC$. This leads to the completion of the proof.

Remark 5.1 A by-product of Proposition 2 is that within the framework for the Proposition we provide a straightforward way to identify the uncovered set. Given that such sets are defined by a little complicated sub-relations, such constructions are not very common.

6 Concluding remarks

In view of our results and discussions (especially in the Remarks) above, perhaps one might remain a little careful in this area: if, given the same primitives for a voting situation, the number of voters changes even by one then a result may not only change (at times drastically) but also, a (somewhat) opposite conclusion may get true. And while we have explored only a few such results in this work, possibility of similar "dramatic" changes may be worth-exploring.

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Appendix: the detailed proofs

Proof of Proposition 1

Before proceeding to the main proof, we establish a supporting Fact and a Lemma.

Note that for any (one-dimensional) straight line segment $L \subset \mathbb{R}^k$, with extreme points denoted by, say, x and z, and a point $y \in L$, one can decompose L into two half-lines xy and yz as follows:
$xy = \{w \in \mathbb{R}^k : w = \beta y + (1-\beta)x\}$ with $0 \leq \beta \leq 1$;
$yz = \{w \in \mathbb{R}^k : w = \beta y + (1-\beta)z\}$ with $0 \leq \beta \leq 1$.

In this regard one can also look up p. 135 of Austen-Smith and Banks (1999).

Further, for the proof of this Proposition we introduce one more piece of notation. For any (one-dimensional) straight line $L \subset \mathbb{R}^k$ and any compact convex subset $A \subseteq Z$, such that $L \cap A$ is non-empty, we denote the unique maximizer of u_i on the compact convex set $L \cap A$ by $\bar{x}_i(L \cap A)$ and call it the *induced ideal point of voter i on the line segment $L \cap A$.*

Fact 1.1 Let $L \subset \mathbb{R}^k$ be any (one-dimensional) straight line. Suppose further that for some compact convex $A \subseteq Z$, $L \cap A \neq \emptyset$. Then, for each $i \in N$, i's preference restricted to $L \cap A$ is single-peaked with $\bar{x}_i(L \cap A)$, the unique maximizer of u_i on the compact convex set $L \cap A$, being the peak.

For proof of this Fact one can see p. 135 of Austen-Smith and Banks (1999).

Lemma 1.1 *Take any voting situation G obeying the conditions of Proposition 1. Suppose x_0 belonging to the interior of Z is the single point in the core and that it is the Condorcet winner as well. Then for any straight line segment $L \subset Z$ with $x_0 \in L$, there exists at least a pair of distinct voters (i, j) such that for $m \in \{i, j\}$, $\bar{x}_m(L)$, the unique maximizer of u_m on the compact convex set L, is x_0.*

Proof of Lemma 1.1 By y and z denote the extreme points of the straight line segment L. Now suppose the statement in the Lemma does not hold. Then, since x_0 is in the core, each of the half-lines yx_0 and zx_0 must contain exactly $|N|/2$ of the induced ideal points on L (as, otherwise, by Fact 1.1 x_0 would be dominated). Consider distinct i and j in N such that
$\bar{x}_i(L) \in yx_0$ and for every $l \in N$ for whom $\bar{x}_l(L) \in yx_0$, $\rho(\bar{x}_l(L, x_0) \geq \rho(\bar{x}_i(L, x_0)$; and
$\bar{x}_j(L) \in zx_0$ and for every $l \in N$ for whom $\bar{x}_l(L) \in zx_0$, $\rho(\bar{x}_l(L), x_0) \geq \rho(\bar{x}_j(L), x_0)$.
Then, by Fact 1.1 above (i.e., for every voter m, the induced preference of m on L is single-peaked with $\bar{x}_m(L)$ being the peak) x_0 cannot dominate any of the points

lying on the line-segment $(\bar{x}_i(L), \bar{x}_j(L))$ contradicting the supposition that x_0 is the Condorcet winner of G. ■

Proof of the main body of Proposition 1

Proof of Part (i) : First we consider a voting situation G where $k = 1$: i.e., Z is a either a single point or a straight line segment. Then, by Lemma 1.1, if x_0 is the Condorcet winner of G then there exists at least a pair of distinct voters (p, q) such that $x_0 = \bar{x}_p = \bar{x}_q$. But that violates the assumption in the part (i) of this Proposition that for no more than one $i \in N$ is it the case that $x_0 = \bar{x}_i$.

Next we consider the case where $k > 1$. Suppose a voting situation G obeys the assumptions for part (i) of Proposition 1 but for that situation x_0, the single point in the core, is the Condorcet winner of G. The proof is done by contradiction.

Call the voter q (if such a voter exists) for whom $\bar{x}_q = x_0$. Note that for any voter $i \in N \setminus \{q\}$, given our assumptions on u_i, $C_i(x_0)$, the upper contour set for i at x_0, is a convex subset (of the full dimension k) of Z (this is ensured by our primitive assumptions that for each $i \in N$, u_i is continuous and strictly concave). Further, again for any $i \in N \setminus \{q\}$, given our assumptions on u_i, by Theorem 3.1 of He and Xu (2013) $C_i(x_0)$ has a unique supporting hyperplane (call it $\sigma_i(x_0)$) at x_0, one of its boundary points.

Take a closed ball $B \subseteq Z$ with x_0 as the centre of B (which is assured as x_0 is in the interior of Z).

For any point x in B call the set of points in B which belong to the straight line connecting x and x_0 and reaching the boundary of B a *diameter* of B (we specify this because "diameter" of a set has been used in literature to define at least one other different concept also): i.e., a diameter through a point $x \in B$ is the intersection of B and the straight line passing through x and x_0. Then

$$B = \cup\{L \mid L \text{ is a diameter of } B \text{ (passing through } x_0)\}.$$

Since x_0 is assumed to be the Condorcet winner of G, by Lemma 1.1 above, for each such diameter L, there exists a pair of distinct voters $(i(L), j(L))$ such that for $m \in \{i(L), j(L)\}$, $\bar{x}_m(L)$, the unique maximizer of u_m on the convex set L is x_0. Therefore, for each diameter L of B, there exists $i \in N \setminus \{q\}$ such that $L \subseteq \sigma_i(x_0)$. Therefore, $B \subseteq \cup_{i \in N\setminus\{q\}} \sigma_i(x_0)$. Recall that for each $i \in N \setminus \{q\}$, $\sigma_i(x_0)$ is the unique supporting hyperplane of $C_i(x_0)$ at x_0 and thus, for each $i \in N \setminus \{q\}$, $\sigma_i(x_0)$ is a subset of at most $(k-1)$ dimension(s). Therefore, since N is a finite set, B, a set of full–k–dimensions, cannot be a subset of $\cup_{i \in N\setminus\{q\}} \sigma_i(x_0)$ which leads to a contradiction.

Proof of part (*ii*) Case 1: Consider the voting situation G_1 as follows.
$N = \{1,2,3,4\}$. Pick the following four points in $\mathbb{R}^2$: $A = (-1,0)$; $B = (0,1/4)$; $C = (1,0)$ $D = (0,-1/4)$. Z is the convex hull of the points A, B, C and D. For each $i \in N$, $\bar{x}_i$ is given as follows. The point $\bar{x}_1 = \bar{x}_2 = (0,0)$; $\bar{x}_3 = (-1,0)$, $\bar{x}_4 = (1,0)$. The voters' preferences are Euclidean.
A look at Figure 2 might help in visualizing this voting situation and in the subsequent arguments.
Obviously the point $O = (0,0)$, in the interior of Z, is in the core of G_1.
Now consider the subset of Z_1 of Z being the triangular area bounded by the straight line segments OA, OB and AB. Take any point x other than O lying on the line segment OA. Then it is straightforward that $O \succ_{\{1,2,4\}} x$. Next, take any point x other than O lying on the line segment OB. Then it is straightforward that $O \succ_{\{1,2,4\}} x$. Finally, take any point $x \in Z_1$ such that x does not lie either on the line segment OA or on the line segment OB. Then consider the triangle ΔxOC. Then the angle $\angle xOC > \pi/2$. But, then by elementary geometry $\rho(x,C) > \rho(O,C)$ and therefore, $O \succ_{\{1,2,4\}} x$. Exactly similarly it can be shown that for every other point $x \in Z \setminus \{O\}$, $O \succ x$.
Case 2: Consider the voting situation G_2 as follows.
$N = \{1,2,3,4\}$. Pick the following four points in $\mathbb{R}^2$: $A = (-1,0)$; $B = (1,0)$; $D = (0,6/5)$ $E = (0,-1/2)$. Z is the convex hull of the points A, B, D and E. For each $i \in N$, $\bar{x}_i$ is given as follows. The point $\bar{x}_1 = \bar{x}_2 = (0,1)$ (call this point C); $\bar{x}_3 = (-1,0)$, $\bar{x}_4 = (1,0)$. The voters' preferences are Euclidean.
A look at Figure 3 might help in visualizing this voting situation and in the subsequent arguments.
Obviously the point $C = (0,1)$, in the interior of Z, is in the core of G_2.
First we show that the core of G_2 is C. First, denote by $Z' \subset Z$, the set of points bounded by the circular arc passing through C with centre at B and the circular arc passing through C with centre at A. It is straightforward that if a point $x \in Z \setminus Z'$ then either $C \succ_{\{1,2,3\}} x$ or $C \succ_{\{1,2,4\}} x$. Next consider the line segment CE which is a subset of Z'. Consider any point $x \neq C$ lying on the segment CE. For x, denote by x_p the point at which the perpendicular from x intersects the line segment CA (the existence of which is assured by our specification of Z). Then $x_p \succ_{\{1,2,3\}} x$. Finally, take any $x \in Z' \setminus CE$ such that x is an element of the subset of Z' bounded by CE and the the circular arc passing through C with centre at B. For x, denote by x_p the point at which the perpendicular from x intersects the line segment CE. Then consider the triangle $\Delta x x_p B$. Then the angle $\angle x x_p B > \pi/2$. But, then by elementary geometry $\rho(x,B) > \rho(x_p,B)$ and therefore, it is easy to see that $x_p \succ_{\{1,2,4\}} x$. Exactly similarly it can be shown that for every other point $x \in Z' \setminus CA$, x is dominated. Therefore, C is the unique point in the core of G_2.
However, it is straightforward that C cannot dominate the point $O = (0,0)$ (as $\rho(A,O) < \rho(A,C)$ and $\rho(B,O) < \rho(B,C)$). Therefore, C is not the Condorcet

winner of G_2. ■

Proof of Corollary 1 Proposition 3.4 in Bhattacharya et al. (2018) shows that for a voting situation G (under conditions even more general than those of Proposition 1 here) UC coincides with a singleton core if and only if the single element in the core is the Condorcet winner of G as well. Therefore, by part (i) of Proposition 1 this Corollary is proved. ■

Proof of Lemma 2.0 Suppose otherwise: i.e., suppose that for no pair of distinct voters i, t is it the case that $C_S(x_0) = C_{it}(x_0)$. Then one of the following two cases has to be true.
Case 1: For some voter $i \in S$, $C_S(x_0) = C_i(x_0)$. Since S contains at least 2 voters, then there must exist voter $j \in S$, $j \neq i$, such that $C_i(x_0) \subseteq C_j(x_0)$. But then $I_i(x_0)$, the indifference curve of i passing through x_0 and $I_j(x_0)$, the indifference curve of j passing through x_0 has a common tangent at x_0. But then $\bar{x}_i$ and $\bar{x}_j$ are collinear–lying on the line segment connecting $x_0, \bar{x}_i$ and $\bar{x}_j$. By A2, $\bar{x}_j$ is in the boundary of Z. But then $\bar{x}_i$ has to be in the interior of Z which would violate A2.
Case 2: Suppose that for no pair of voters i, t in S is it true that $C_S(x_0) = C_{it}(x_0)$ but, instead, $C_S(x_0) = C_i(x_0) \cap C_t(x_0) \cap C_m(x_0) \cap \cdots$ for some voters $i, t, m, \cdots$ (in S) etc. But this is not possible unless for at least one voter (say, m) $I_m(x_0)$, the indifference curve of m passing through x_0, intersects either $I_i(x_0)$ or $I_t(x_0)$ at a point other than x_0. But then A5 is violated. ■

Proof of Proposition 2

We start with the following elementary but useful Lemma.

Lemma 2.1 *Suppose a coalition S is in P. Then, if $i \in S$ then $j \notin S$ where $j = \omega(i)$.*

Proof of Lemma 2.1 Suppose not: i.e., suppose there exist voters $i, j \in S$ such that $j = \omega(i)$. By definition of P, there exists policy $y \in C_S(x_0)$ such that $y \neq x_0$. By definition of $C_S(x_0)$, $u_i(y) \geq u_i(x_0)$ and $u_j(y) \geq u_j(x_0)$. But then, by strict concavity of the pay-off functions, for some policy z belonging to the line segment yx_0, $u_i(z) > u_i(x_0)$ and $u_j(z) > u_j(x_0)$. But that, then, violates the fact that $x_0 \in K_{ij}$. ■

Proof of Step 1

The proof proceeds in two sub-steps.
In Sub-step 2.1 we show the following. Take any $x \in C_i(x_0) \setminus \{x_0\}$. Take the straight line segment xx_0 and extend it to the boundary of Z. Call this extended line segment l. Then $l \cap (C_j(x_0) \setminus \{x_0\}) \neq \emptyset$. To see this, suppose otherwise. Then

$l \cap (C_j(x_0)$ is the single point x_0. But then, by A1 above (i.e., the preferences being Euclidean), l is the tangent at x_0 to the arc $\overset{\frown}{\bar{x}_j x_0}$. But then, since $(i,j) \in \Omega$, l is the tangent at x_0 to the arc $\overset{\frown}{\bar{x}_i x_0}$ too. But that contradicts the supposition that $x \in C_i(x_0) \setminus \{x_0\}$.
Next, note that by Lemma 2.0 and Lemma 2.1, $C_S(x_0) = C_{it}(x_0) = (C_i(x_0) \cap C_t(x_0))$ has a non-empty interior.
Next, by Sub-step 2.1, if $C_S(x_0)$ has a non-empty interior then $C_{N\setminus S}(x_0)$ also has a non-empty interior. Now suppose $C_{N\setminus S}(x_0) = C_{mp}(x_0)$ (with $m, p \in N \setminus S$) where, without loss of generality, voter m is different from either j or w. Then the arc $\overset{\frown}{\bar{x}_m x_0}$ has a non-empty intersection with the interior of $C_S(x_0)$ (i.e., $C_{it}(x_0)$) because voter m is neither j nor w and thus, the arc $\overset{\frown}{\bar{x}_m x_0}$ is not tangential to either i's or t's indifference curve through x_0. Note that by Lemma 2.1, the voter $\omega(m)$ is in S. But, then there exists some $x \in C_S(x_0)$ such that $u_m(x) > u_m(x_0)$ and $u_{\omega(m)}(x) > u_{\omega(m)}(x_0)$ as well. But that leads to a contradiction as then x_0 cannot lie on $K_{\{m,\omega(m)\}}$. Thus, $C_{N\setminus S}(x_0) = C_{jw}(x_0)$. ■

Proof of Step 2

To prove this Step we first establish the following Lemma.

Lemma 2.2 If $q \in S$ (resp. $N \setminus S$) then $\bar{x}_q$ lies on the subset of $bd(Z)$ bounded by the line segments $x_0\bar{x}_t$ and $x_0\bar{x}_i$ (resp. $x_0\bar{x}_w$ and $x_0\bar{x}_j$).

Proof of Lemma 2.2 Suppose not and without loss of generality, consider the possible case of voter $q \in S$ for whom $\bar{x}_q$ lies, say, on on the subset of $bd(Z)$ bounded by the line segments $x_0\bar{x}_w$ and $x_0\bar{x}_i$ (looking up at Figure 6 might be helpful here). Draw the arc $\overset{\frown}{\bar{x}_q x_0}$ centred at $\bar{x}_q$ passing through x_0. Recall the (elementary) fact that the line segment $\bar{x}_q x_0$ gives the direction of the normal of the arc $\overset{\frown}{\bar{x}_q x_0}$ at the point x_0. But then a subset of the arc $\overset{\frown}{\bar{x}_q x_0}$ is a subset of $C_{it}(x_0)$ which contradicts our finding in Lemma 2.0 above that $C_S(x_0) = C_{it}(x_0)$. The argument for the other possible cases are exactly similar. ■

Proof of the remaining part of Step 2 Note that by A4 must exist a pair of voters $(m,p) \in \Omega$ such that the line segments K_{tw} and K_{mp} are perpendicular to each other at x_0 : i.e., K_{mp} has to be tangential to $I_t(x_0)$ at x_0. By Lemma 2.1, one of m and p, say m, is in S. But if $\bar{x}_i \in C_t(x_0)$ then, by Lemma 2.2 and A5, $\bar{x}_m \in C_t(x_0)$ as well. But then any straight line passing through $\bar{x}_m$ and x_0 cannot be tangent to $I_t(x_0)$ at x_0. ■

Proof of Step 3

First we establish the following Lemma.

Lemma 2.3 *For any pair $(m, p) \in \Omega$, take any K_{mp}, the mp-core straight line segment. Take $x \in Z \setminus K_{mp}$ and let r_x be the perpendicular line segment from x to K_{mp} (or its extension in $\mathbb{R}^2$). Then for any point y lying on r_x distinct from x, there is a subset of voters $Y \subseteq N$ such that $|Y| > |N|/2$ and for each $i \in Y$, $\rho(\bar{x}_i, y) < \rho(\bar{x}_i, x)$.*

Proof of Lemma 2.3 Note that, especially by A2, K_{mp} divides Z into two "half-sets" (Figure 7 provides an illustrative example for the argument here). Suppose x lies in one of these half-sets Z''. By the Assumption A4 above, for $(|N| - 2)/2$ of the voters' ideal points lie in the other half-set Z'. Define Y to be the union of such $(|N|-2)/2$ voters and $\{m, p\}$. Consider any $i \in Y$ and the angle $\angle \bar{x}_i y x$. Then $\pi/2 < \angle \bar{x}_i y x \leq \pi$. Then, by plane geometry $\rho(\bar{x}_i, y) < \rho(\bar{x}_i, x)$. ∎

Next we proceed along the following three sub-steps.

Sub-step 3.1: In this sub-step we consider the case where K_{ij} and K_{wt} are perpendicular to each other at x_0.
Parition $Z \setminus (E_y \cup K_{ij} \cup K_{wt})$ into the following nine subsets (please refer to Figure 5 as an illustration and please note that positioning of, say, $\bar{x}_i$ is completely arbitrary):
$R1 :=$ the area bounded by I_t^- and the straight line segment $x_0\bar{x}_t$;
$R2 :=$ the area bounded by I_t^- and the straight line segment $x_0\bar{x}_j$;
$R3 :=$ the area bounded by I_w^+ and the straight line segment $x_0\bar{x}_j$;
$R4 :=$ the area bounded by I_w^+ and I_j^+;
$R5 :=$ the area bounded by I_j^+ and the straight line segment $x_0\bar{x}_w$;
$R6 :=$ the area bounded by I_i^- and the straight line segment $x_0\bar{x}_w$;
$R7 :=$ the area bounded by I_i^- and the straight line segment $x_0\bar{x}_i$;
$R8 :=$ the area bounded by the curve l_y and the straight line segment $x_0\bar{x}_i$;
$R9 :=$ the area bounded by the curve l_y and the straight line segment $x_0\bar{x}_t$.

Now, consider any $x \in R1$. Drop the perpendicular line from x to K_{ij} (or its extension into $\mathbb{R}^2$). Then this perpendicular must intersect I_t^- at some point z. Then by Lemma 2.3 above, $z \succ x$.
Next consider $x \in R9$. Drop the perpendicular line from x to K_{ij} (or its extension into $\mathbb{R}^2$). Then this perpendicular must intersect l_y at some point z. Then by Lemma 2.3 above, $z \succ x$.
For the subsets $R2$ to $R8$ also, for any point x in these subsets it can be shown in similar ways that there exists $z \in E_y$ such that $z \succ x$.

Sub-step 3.2: In this sub-step we consider the case where K_{ij} and K_{wt} are not

perpendicular to each other at x_0.
Then, for the pair (i, j) (respectively (t, w)), already defined above in Step 1, consider the pair (i', j') (respectively (t', w')) such that K_{ij} and $K_{i'j'}$ (respectively K_{tw} and $K_{t'w'}$) are perpendicular to each other at the core point x_0 (which is assured by A4 above). Then, mimicking the method in Sub-step 1 above it can be shown that for any $x \in Z \setminus (K_{ij} \cup K_{i'j'} \cup K_{tw} \cup K_{t'w'})$ there exists $z \in E_y$ such that $z \succ x$.

Sub-step 3.3: Finally consider any mp-core K_{mp} such that $(m, p) \in \Omega$. Consider $K_{m'p'}$, the $m'p'$-core perpendicular to K_{mp} at x_0. Then, by mimicking the proof of Lemma 2.3 above we can show that x_0 dominates each $x \in K_{mp} \setminus \{x_0\}$.

Thus, E_y is externally stable. ■

Proof of Step 4

Suppose, for some $y \in C_S(x_0)$, such a z exists on I_j^+ (the argument for I_w^+ would be identical) for which $z \succ y$. Then there must exist $p \in S$ such that $u_p(z) > u_p(y)$ (because $N \setminus S$ has only $|N|/2$ voters). Since $u_p(y) \geq u_p(x_0)$ (as $y \in C_S(x_0)$), $u_p(z) > u_p(x_0)$. Since, by Step 1, I_j^+ is in the boundary of $C_{N \setminus S}(x_0)$, for each $q \in N \setminus S$, $u_q(z) \geq u_q(x_0)$. But, as the pay-off function of each voter is strictly concave, then there exists a point v lying on the line segment zx_0 for which $u_m(v) > u_m(x_0)$ for each $m \in \{p\} \cup (N \setminus S)$. But then x_0 cannot be in the core.
Next suppose such a z exists on I_i^- (the argument for I_t^- would be identical). Then there must exist $p \in S$; $p \neq i$; such that $u_p(z) > u_p(y)$ (because $N \setminus S$ has only $|N|/2$ voters). By Assumption A5 the arcs $\overset{\frown}{\bar{x}_p x_0}$ and $\overset{\frown}{\bar{x}_i x_0}$ intersect only at the single point x_0. Thus, $z \notin C_p(x_0)$: i.e., $u_p(x_0) > u_p(z)$. But since $u_p(y) \geq u_p(x_0)$ (as $y \in C_S(x_0)$) this leads to a contradiction. ■

Proof of Step 5

We prove Step 5 through the following sub-steps.

Sub-step 5.1: In this sub-step first we establish another elementary but helpful Lemma.

Lemma 2.4 Denote by p_S^B the point at which p_S, the perpendicular dropped on K_{it} from x_0 intersects K_{it}. For every $q \in S$ (resp. every $q \in N \setminus S$) the angle $\angle \bar{x}_q x_0 p_S^B$ is such that $0 \leq \angle \bar{x}_q x_0 p_S^B \leq \pi/2$ (resp. $\pi/2 \leq \angle \bar{x}_q x_0 p_S^B \leq \pi$).

Proof of Lemma 2.4 Since $x_0 \notin K_{it}$ (by Lemma 2.1), $\angle \bar{x}_i x_0 p_S^B < \pi/2$ (Figure 6 provides a visualization). Consider any $q \in S \setminus \{i\}$ such that $\bar{x}_q$ lies on the boundary

of Z bounded by the point at which the extension of p_S intersects $bd(Z)$ and $\bar{x}_i$ (recall that by Lemma 2.2, for any such voter q, $\bar{x}_q$ lies on on the subset of $bd(Z)$ bounded by the line segments $x_0\bar{x}_t$ and $x_0\bar{x}_i$). Note that by A2, $\bar{x}_q$, $\bar{x}_i$ and x_0 cannot be collinear (because $\bar{x}_j$, $\bar{x}_i$ and x_0 are already on the same straight line). Furthermore, since Z is convex, $\angle\bar{x}_q x_0 p_S^B < \angle\bar{x}_i x_0 p_S^B < \pi/2$.
For each of the other cases (for example, when $q \in S$ is such that $\bar{x}_q$ lies on the boundary of Z bounded by the point at which the extension of p_S intersects $bd(Z)$ and $\bar{x}_t$ etc) the argument is exactly similar. ■

Sub-step 5.2: Next, in this and the next sub-step we prove the following Claim.

Claim: Without loss of generality, consider any $y \in A_i$ (recall, if necessary, the definition of this set from the discussion following Step 2 in the main body of the paper) and consider points $x, z \in \gamma_y$ such that $\lambda_y^{-1}(x) < \lambda_y^{-1}(z)$ (that is, in reference to the expository Figure 6, the point z is "below" point x "along" γ_y).[3] Then:
(i) for every $q \in S \setminus \{i, t\}$, $u_q(x) < u_q(z)$ and for $q \in \{i, t\}$, $u_q(x) \leq u_q(z)$;
(ii) for every $q \in N \setminus S$, $u_q(x) > u_q(z)$.

In this Sub-step we take up the case for which both x, z are in p_y^T.

In this case, without loss of generality, pick the sub-case for which $q \in N \setminus S$ and then consider the triangle $\Delta\bar{x}_q xz$. The angle $\angle\bar{x}_q xz \geq \angle\bar{x}_q x_0 z > \pi/2$ (by Lemma 2.4). Then, by elementary geometry (of planes in two-dimensions) the side $\bar{x}_q x$ is shorter than the side $\bar{x}_q z$. So, for every $q \in N \setminus S$, $u_q(x) > u_q(z)$.

The argument for the other possible sub-cases (e.g., with voter q in S etc) is exactly similar.

Sub-step 5.3: Continuing with the issue addressed in Sub-step 5.2 above, next we consider the case where both x, z are in α_y.

For this case, we break down our consideration into four sub-cases.

Drop a perpendicular from the point x (resp. z) on K_{it} and let this intersect $bd(Z)$ at points p_x and p'_x (resp. p_z and p'_z) (please refer to Figure 6 for any needed help in visualization).

Sub-case (i): In this sub-case consider the scenario where $q \in S \setminus \{i, t\}$ such that $\bar{x}_q$ lies in the subset of $bd(Z)$ bounded by p_z (please refer again, if needed, to Figure 6 for visualization) and $\bar{x}_i$. Note that, by the specification of the points x and z, $\angle\bar{x}_i xz = \angle\bar{x}_i zx$. Therefore, given A2, Lemma 2.2 above (and Z being a convex

[3] Recall from the outline of this Step given in the main body of the paper that $\gamma_y = p_y^T \cup \alpha_y$.

set), $\angle\bar{x}_q xz < \angle\bar{x}_i xz$ and $\angle\bar{x}_q zx > \angle\bar{x}_i zx$. Therefore, $\angle\bar{x}_q zx > \angle\bar{x}_q xz$ and thus, considering the triangle $\Delta\bar{x}_q zx$, $\rho(\bar{x}_q, x) > \rho(\bar{x}_q, z)$.

Sub-case (*ii*): In this sub-case consider the scenario where $q \in N \setminus S$ such that $\bar{x}_q$ lies in the subset of $bd(Z)$ bounded by p'_x (please refer again, if needed, to Figure 6 for visualization) and $\bar{x}_j$. Note that then $\angle\bar{x}_j xz > \pi/2$. To see this, consider the angle $\angle\bar{x}_j x'z$ where x' is the point at which the extension of the straight line segment zx intersects the straight line segment K_{ij} (again, refer to Figure 6 for visualization). Since zx does not lie on the tangent to the arc $I_i(y)$ at which this arc intersects K_{ij}, $\angle\bar{x}_j x'z > \pi/2$ and so, $\angle\bar{x}_j xz > \pi/2$. Therefore, $\angle\bar{x}_j xz > \angle\bar{x}_j zx$. Therefore, given Lemma 2.2 (and Z being a convex set), $\angle\bar{x}_q xz > \angle\bar{x}_q zx$. Then, considering the triangle $\Delta\bar{x}_q zx$, $\rho(\bar{x}_q, x) < \rho(\bar{x}_q, z)$.

Sub-case (*iii*): In this sub-case consider the scenario where $q \in S \setminus \{i\}$ such that $\bar{x}_q$ lies in the subset of $bd(Z)$ bounded by p_z (please refer again, if needed, to Figure 6 for visualization) and $\bar{x}_t$. Then draw a straight line through $\bar{x}_q$ parallel to K_{it} and consider the right-angled triangle with $\bar{x}_q$, x and the point at which that line intersects the line $p_x p'_x$ being the three vertices. Similarly, consider the right-angled triangle with $\bar{x}_q$, z and the point at which that line intersects the line $p_z p'_z$ being the three vertices. Then the length of the hypotenuse of the first triangle, $\rho(\bar{x}_q, x)$ is clearly greater than that for the second triangle, $\rho(\bar{x}_q, z)$.

Sub-case (*iv*): In this final sub-case consider the scenario where $q \in N \setminus S$ such that $\bar{x}_q$ lies in the subset of $bd(Z)$ bounded by p'_x (please refer again, if needed, to Figure 6 for visualization) and $\bar{x}_w$. Mimicking the method of proving Sub-case (*iii*) above (i.e., drawing a straight line through $\bar{x}_q$ parallel to K_{it} etc) it is clear that $\rho(\bar{x}_q, x) < \rho(\bar{x}_q, z)$.

Further, note that if $\alpha_y \subseteq C_i(x_0)$ (resp. $C_t(x_0)$) then, obviously, $u_i(x) = u_i(z)$ (resp. $u_t(x) = u_t(z)$).

Thus, the Claim is proved.

Sub-step 5.4: Finally we consider the point B^y (for recalling please refer to Figure 4) and any $z \in p_y^B \setminus \{B^y\}$. Then, by an argument exactly similar to that for Sub-step 5.2 above, for each voter q in the majority coalition $(N \setminus S) \cup \{i, t\}$, $u_q(B^y) > u_q(z)$. Therefore, for no $z \in p_y^B$ is it true that for some $x \in \alpha_y$, $z \succ x$. Using a similar logic repeatedly, for example, for $z \in p_y^B$ with respect to some $x \in P_y^T$ etc, Step 5 is proved. ■

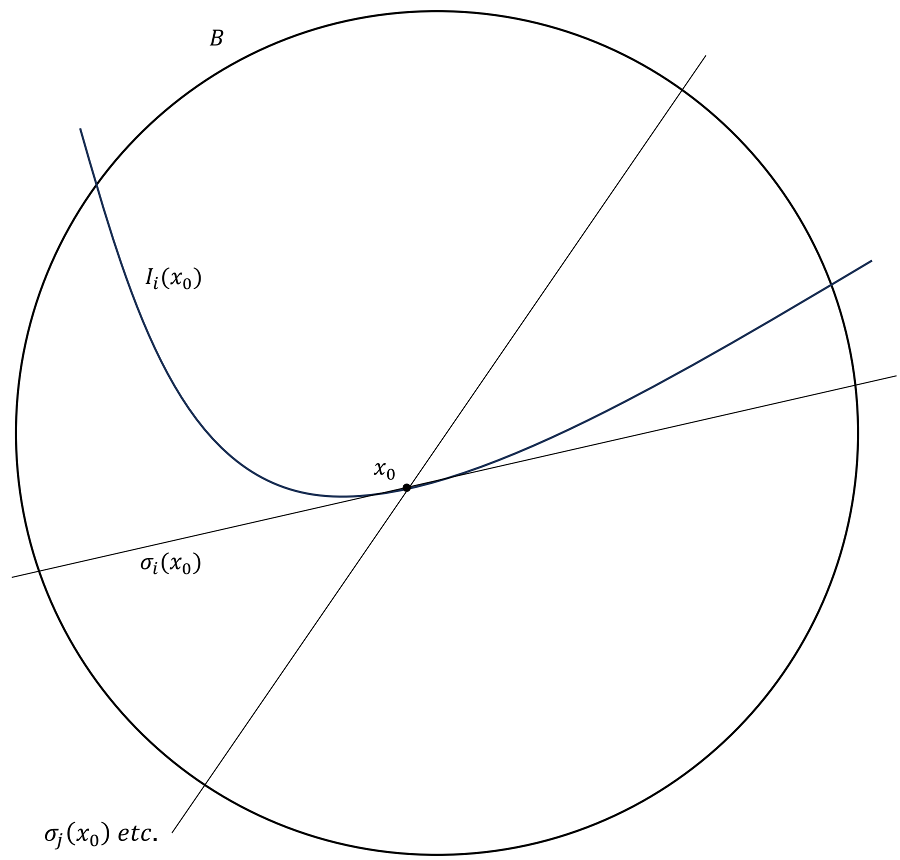
Proof of Step 6

Note that $A_S = \cup_{y \in A_S} \gamma_y$. Pick any $y \in A_S$. Then, by Step 3 above, E_y is externally stable. Next, by Steps 4 and 5 above, for no $x \in \gamma_y$ and no $z \in E_y$ is it the case that $z \succ x$. Therefore, by Proposition 37 in Duggan (2013), for each $y \in A_S$, γ_y is a subset of UC. Therefore, $A_S \subseteq UC$.
Therefore, for each coalition $T \in P$, A_T (defined in analogy to that for coalition S) is a subset of UC. Since for each $T \in P$, A_T has a non-empty interior and x_0 is in the interior of Z, there must exist a ball B of some radius $\delta > 0$ with centre at x_0 for which $B \cap M(x_0)$ is a subset of the (Gillies) uncovered set UC.

Thus, part (i) of the Proposition is proved. ∎

And part (ii) of the Proposition is straightforward because with A6 additionally, for every $S \in P$, $A_S = C_S(x_0)$. ∎

Figures



$Figure$ 1: Illlustrating (for two dimensions) the argument for proving part (i) of Proposition 1

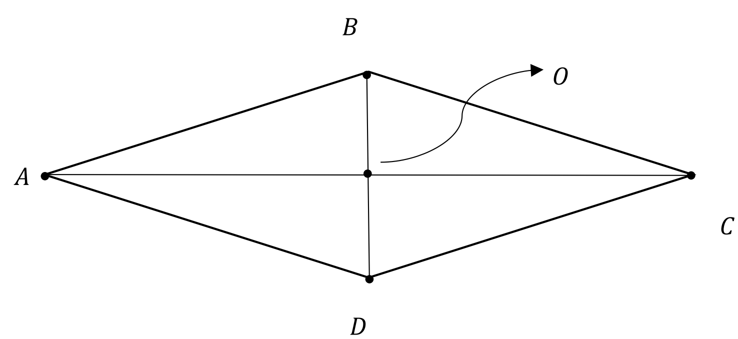


Figure 2: A visual aid for Case 1 of the proof of part (ii) of Proposition 1

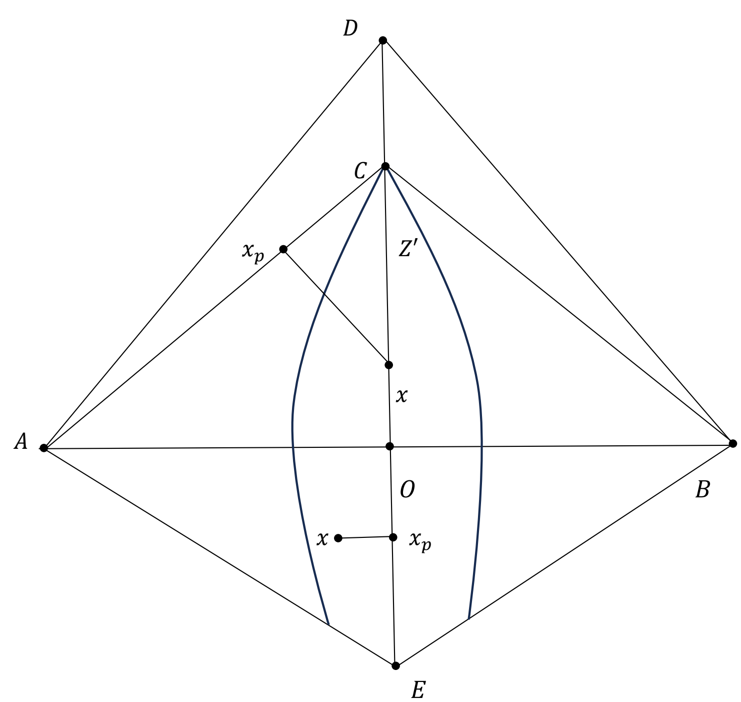


Figure 3: *A visual aid for the proof of Case* 2 *of part* (*ii*) *of Proposition* 1

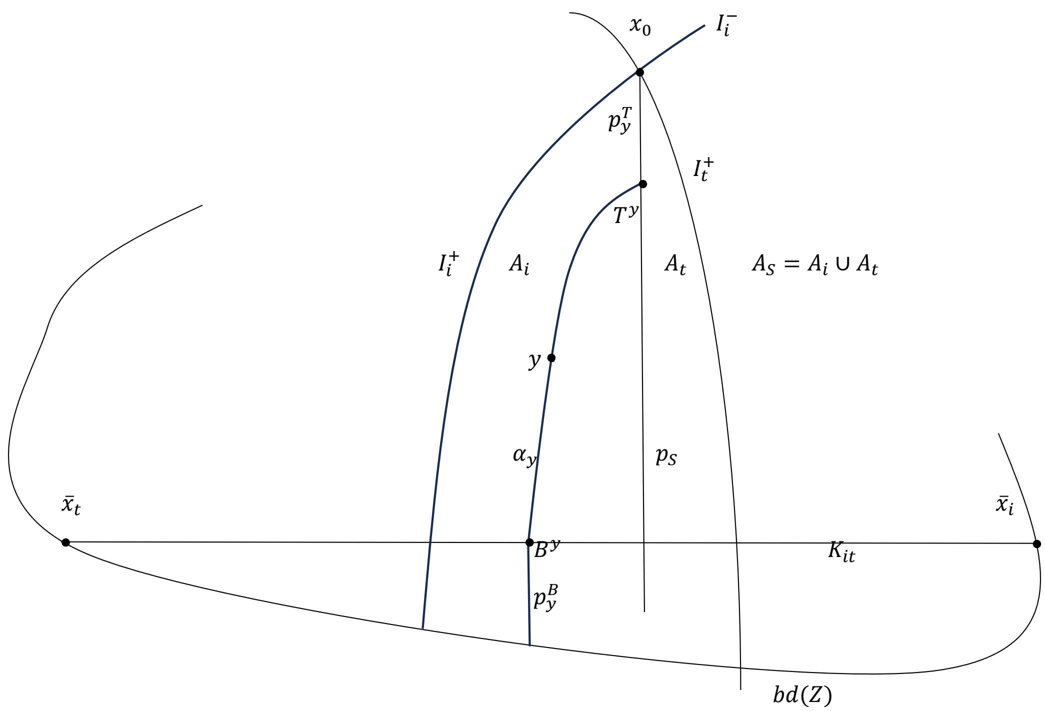
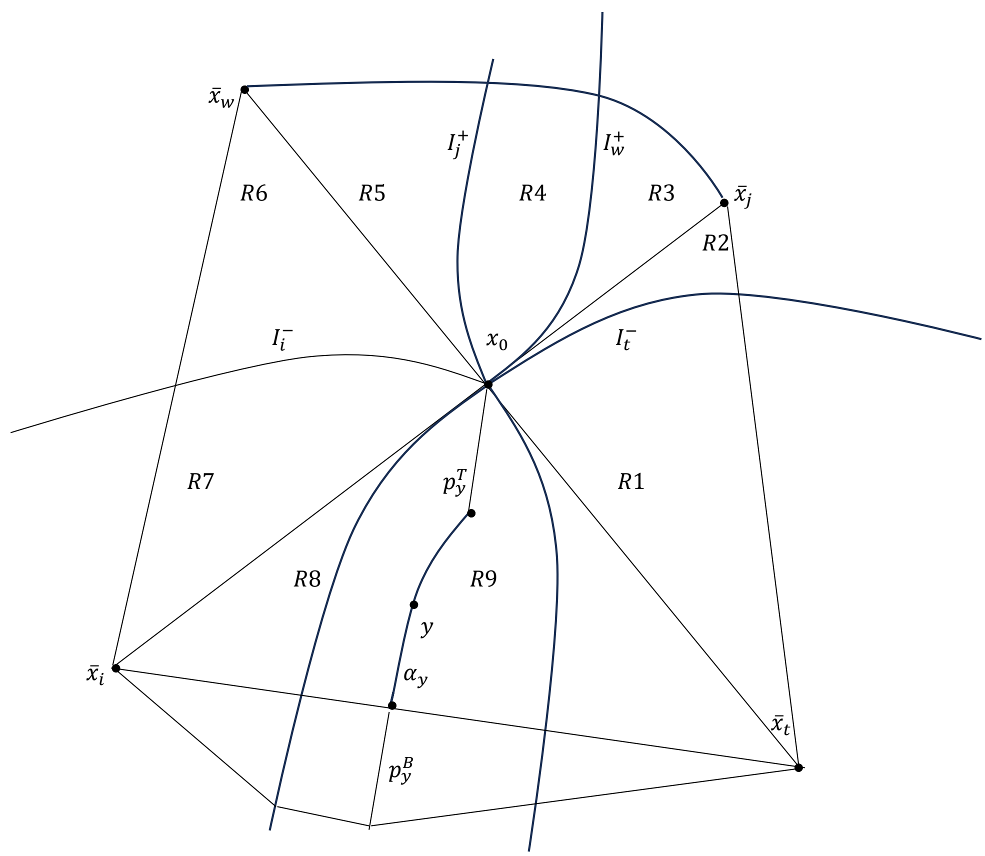


Figure 4: An illustration of the set l_y as well as some other relevant subsets of it



$l_y = p_y^T \cup \alpha_y \cup p_y^B;\ E_y = I_i^- \cup I_t^- \cup I_j^+ \cup I_w^+ \cup l_y$

Figure 5: *An aid for visualizing Step* 3 *of the proof of Proposition* 2

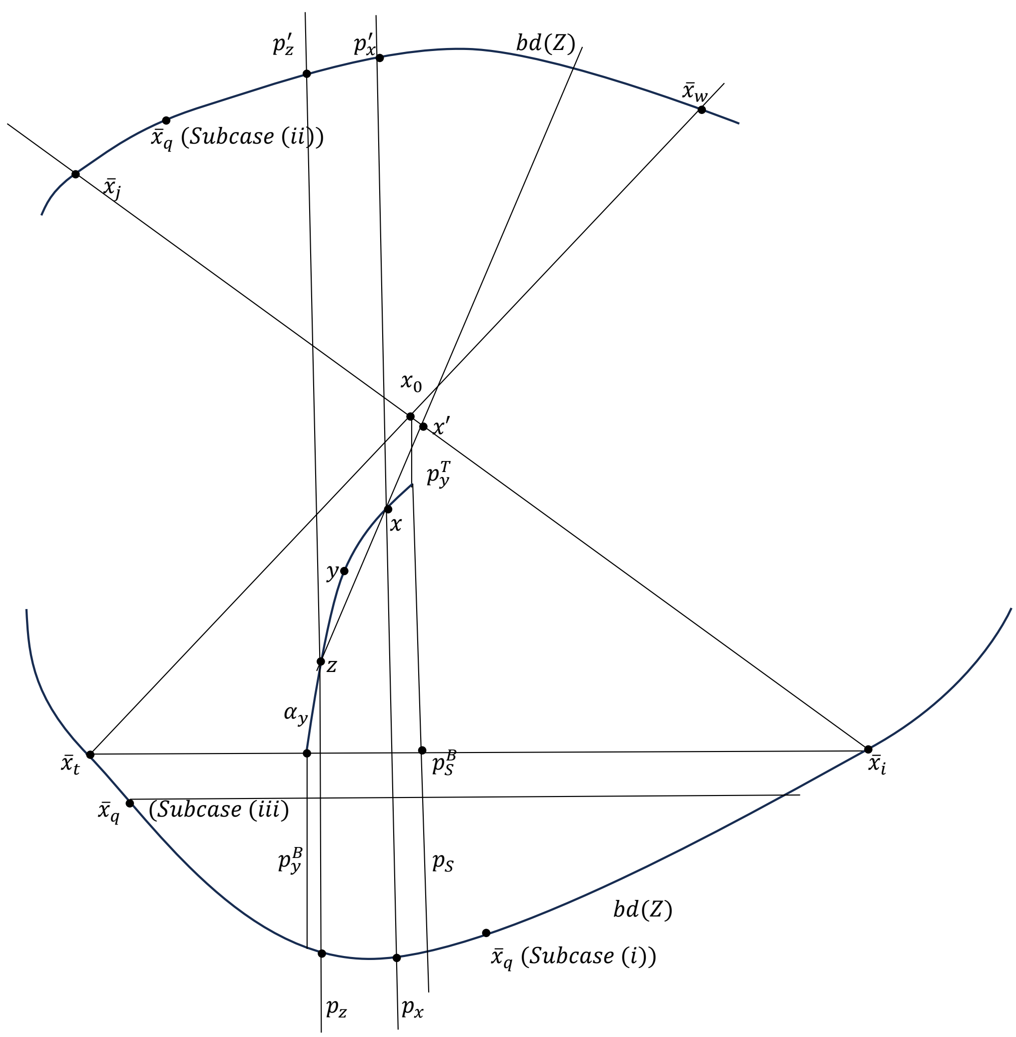


Figure 6: An aid for visualizing Step 5 of the proof of Proposition 2

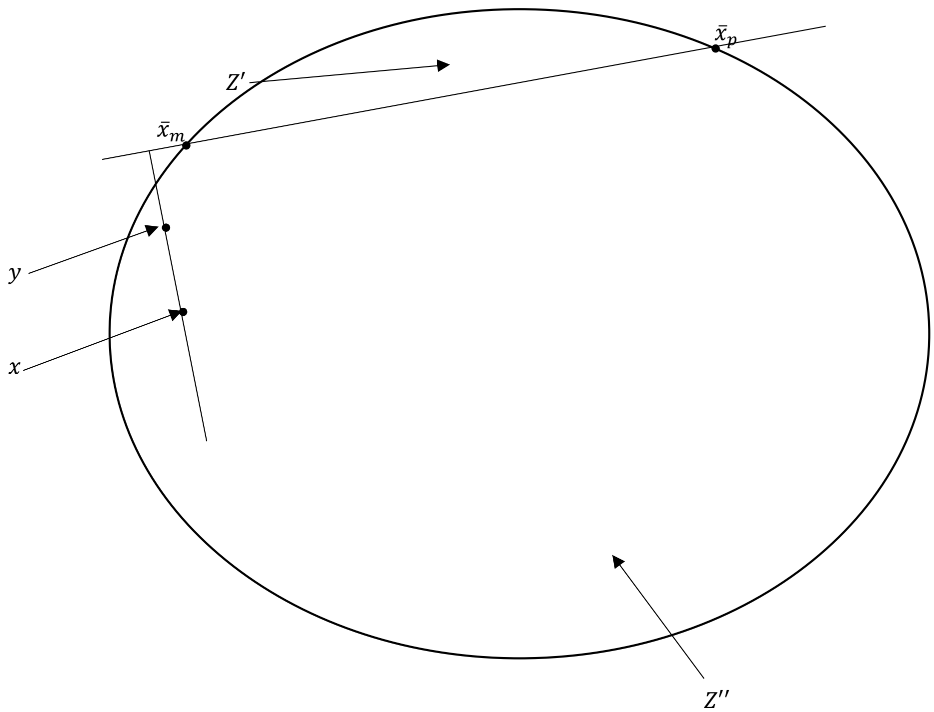


Figure 7: *An illustration with respect to Lemma* 2.3