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Optimal Differentiation and Spatial Competition: The Spokes Model with Product Delivery

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Abstract

The spokes model is a recent framework to study n-firms spatial competition. In a spatial framework firms delivering their product can price discriminate with respect to consumers' location. Conditions for the existence of a price-location equilibrium of the spokes model with delivered product are established in both the case where there are as many firms as spokes and in the case not all spokes are occupied. The equilibrium outcome may be interpreted as one firm supplying a "general purpose product" while others focusing on their "niche".

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1 Introduction

This paper analyzes endogenous location in the spokes model when firms are allowed to price discriminate. It also establishes the existence and properties of a price-location equilibrium in this context.

The spokes model is a theoretical framework to model non-localized spatial competition between n firms. Firms and customers are located over spokes of constant length which have a common centre. Consumers can buy from whichever firm they like but if the firm is not located on their own spoke, either the customer or the delivering firm has to travel through the centre of the market. In this sense the spokes model can be seen as a natural extension of Hotelling competition on a line, when there is a generic number n of firms in the market. The spokes spatial configuration has been recently introduced in the literature on product differentiation by Chen-Riordan[3]. Two are the main contributions of Chen-Riordan's paper: the first is to prove that in a limiting equilibrium as the number of spokes and firms tend to infinity the spokes model captures Chamberlin's original idea of monopolistic competition; the second is to highlight that strategic interaction in the spokes model may imply price increasing competition. Although the features of the spokes model may not perfectly match the ones of real world markets[1], the framework can be considered as an important theoretical alternative to the circular city model model (Salop[17]) when the neighbouring effects of competition are not particularly relevant. The new but growing literature[2] on the spokes model has focused on price and entry choices of firms exogenously

[1] One notable exception may be the concrete sector (Syverson[18]). Concrete is produced by firms at several locations in the territory and it is usually shipped to final users. This market is often cited in discussions of spatial price discrimination (Phlips[16]) and of the spokes model (Chen-Riordan[4]).

[2] The spokes model is used, between others, by Caminal-Claici[1] in the context of loyalty rewarding schemes, by Caminal-Granero[2] to study the provision of quality by multi-product firms, by Ganuza-Hauk[7] to address allocation of ideas in tournaments, by Chen-Riordan[4] regarding vertical integration.

located at the extreme of their own spoke.

Product delivery by firms is a very common assumption in the literature on spatial pricing. The analysis of location-contingent pricing can be traced back to Greenhut-Ohta[9] in the monopolistic case and Greenhut-Greenhut[8] in an imperfectly competitive case. Perfect price discrimination is more likely in a spatial setting than in most other market configurations, as firms can easily get precise information on the address of the customers and deliver them the products. As a consequence, there is an extensive literature on optimal firms' location and product delivery. Thisse-Vives[19] were the first to point out that in a spatial context competitive price discrimination makes all firms worse off. While a monopolist can extract all of consumers' rent by discriminating with respect to location, in a duopoly with perfect information firms will match the opponent's offer to a given customer located at a generic point. Lederer-Hurter[14] establish the existence of a price-location equilibrium in a duopolistic spatial framework. They prove under very general assumptions that the profit maximizing location chosen by firms correspond to the socially optimal one. MacLeod-Norman-Thisse[15] consider an n-firms spatial model and prove the existence of a price-location equilibrium with free-entry. Their assumptions on the spatial configuration is compatible with a circular city model *à la Salop*. The conclusions are similar to the ones of Lederer-Hurter[14], although free entry might determine a too large or a too small number of varieties.

In this paper the topological structure of the spokes model is unmodified but a different game is considered. Firms are allowed to price discriminate customers with respect to their location on the spokes by delivering the product to the consumers' address[3]. Endogenous location in an n firm market with no neighbouring effects is tackled. The game analyzed allows

[3] The opposite assumption of consumers traveling to the firm's outlet would leave the results unaffected (Vogel[20]). However, assumptions on perfect information and on the impossibility of arbitrage may be less convincing in that case.

customers to choose whichever brand/firm they like of the n existing on the market. This feature is in sharp contrast with the spokes model with uniform pricing: in that case tractability requires a given customer to have positive demand for *at most* two brands. The analysis of this game allows to shed light on a number of the issues discussed above. This paper not only extends Theorem 1 and Theorem 3 in Lederer-Hurter[14] in the context of n firms one-dimensional competition but also establishes the conditions for the existence of a price-location equilibrium when the number of spokes is greater than the number of firms on the market. The properties of this new equilibrium are characterized, highlighting the differences with respect to the benchmark case. As expected, full competition at each location still drives down prices: this is reflected in a sharp decrease in firms' profits, as compared to the uniform price case. The main result, however, is to prove that when there exist parts of the markets not covered by firms, an asymmetric outcome arises. One firm locates in the middle of the market, while the others concentrate on serving their own spoke. This can be interpreted as one firm supplying a sort of "general purpose" product while all others targeting the market's "niches". The multiplicity of equilibria is not occurring if one of the firms has a cost advantage. In that case, the most efficient firm is the natural candidate for supplying the general purpose product. The social optimality of product delivery in the short run seems robust to the presence of n firms on the market.

The rest of the paper is structured as follows. Section 2 briefly introduces the spokes model and its main features when firms are allowed to price discriminate. Section 3 describes the game analyzed and provides the main results. The existence and properties of a price-location equilibrium are presented both for the case in which there are as many firms and spokes and for the case in which the number of spokes is greater than the one of firms. Mixed strategies are also considered. Section 4 concludes. Unless otherwise stated, the proofs of all propositions can be found in the Appendix.

2 Price Discrimination in the Spokes Model

The market is described as a set of spokes with a common core. Each spoke has a constant length which is normalized to $l_s = 1/2$, $s = 1...N$. The market is constituted of a fixed number of spokes N. Customers are distributed along each spoke according to a distribution function $f(x_s)$. Without loss of generality it will be assumed throughout the paper that customers are uniformly distributed over the market, so that at each point of a spoke there is a density $f(x_s) = 2/N$ of customers; any non degenerate distribution function can be employed without affecting the results[4]. Each customer has a valuation v of each unit of the good. She demands one unit of the good from firm i if: $v - p_i > 0$. If no firm can provide a positive utility the customer stays out of the market: $v - p_i \leq 0 \ \forall i = 1...n$: this possibility is ruled out and it is assumed that v is high enough for the market to be covered.

On the supply side, it is assumed that $n \leq N$ firms locate over the spokes. Each spoke is occupied at most by one firm[5]: this feature implies that each firm has its own spoke but, if the inequality holds strictly, not on all spokes there is a firm. The good supplied is *a priori* homogeneous: the only source of differentiation is given by the distance that separates the consumers from the firm. Unlike Chen-Riordan [3], who introduced the spokes model, it is not necessary to assume that each consumer has only one favourite brand as an alternative to the one represented by his own spoke. Competition can take place between all firms at the same time. The entry stage is overlooked: assuming an exogenously given number of firms enter the market does not

[4]The intuition for the result is the following: in computing both firms' profits and social cost functions, each location has to be considered independently. This is due to the assumption of product delivery, which allows firms to condition the price schedule to consumers' location. Local competition implies that the shape of the distribution function affects the optimal location but does not affect the properties of it.

[5]This assumption is not strictly necessary: when $n \leq N$ it can be shown that no pair of firms have incentive to locate on the same spoke.

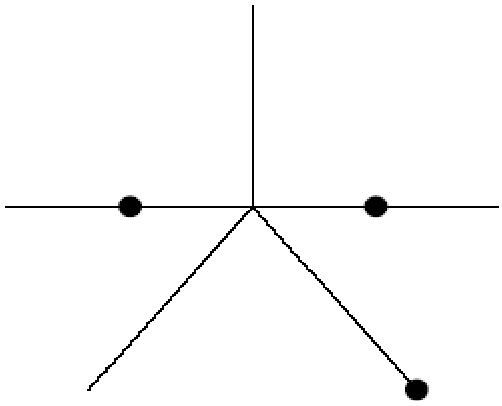

Figure 1: Spokes model with endogenous location with $n = 3$ and $N = 5$.

prevent to capture the main insights of the analysis. The focus of this paper is on the location choice of firms. A generic firm i can locate on whatever point of its spoke l_i which is denoted by y_i. This feature implies that $y_i \in [0; 1/2]$[6].

As firms deliver their product to consumers, they are allowed to price discriminate customers according to their location over the spokes. A generic customer located on a spoke s is identified by x: consumers in $x = 0$ are located at the extreme of the considered spoke while consumers at $x = 1/2$ are exactly at the center of the market. The location of each consumer, however, is spoke dependent: a consumer is fully identified by x_s although at times it will be convenient to denote x_f, $f = 1...n$, for consumers on spokes occupied by firms and x_e, $e = 1...N - n$ for consumers located on empty spokes.

The assumptions stated imply that competition between firms takes place for each individual customers at any specific given location. Figure 1 illustrates the spokes model in case two firms are located in the interior of their

[6] Chen-Riordan[3] and most of the following literature assumed that firms were all located at the origin of each spoke, i.e. $y_i = 0 \ \ \forall i = 1..n$.

spoke while one is at the extreme. The two remaining spokes are not occupied by firms, but consumers are uniformly distributed all over the spokes. The topology of the spokes model is not trivial: for this reason a discussion of the definition of *distance* is in order. The distance between firm i, located at y_i and the customer located at x_s is defined as $d(y_i, x_s)$. The notion of distance, then, is also *spokes-dependent*. In particular, if the firm and the customer are both located on the i-th spoke, then distance can be written as:

$$d(y_i, x_s) = |y_i - x_s| \qquad s = i$$

But if the firm is located on a different spoke with respect to customer $x(l)$, then distance is written as:

$$d(y_i, x_s) = \left(\frac{1}{2} - y_i\right) + \left(\frac{1}{2} - x_s\right) = 1 - y_i - x_s \qquad \forall s \neq i$$

as the firms always have to travel towards the center of the market to deliver the product to consumers located over different segments.

Consistent with most of the literature on spatial price discrimination, it is assumed that serving a customer has a cost which is proportional to distance: the unit transportation cost is identical for all firms and all customers and is denoted by t. Each firm can produce the good through a technology characterized by a unit and marginal cost of c_i.

3 A Price-Location Equilibrium of the Spokes Model

3.1 The Game

It is assumed that $n \leq N$ firms have entered the market, as the entry stage is not explicitly modelled. The logical sequence of the game is as follows:

1. Nature has assigned to each of the n firm one and one only spoke, between the N available.

2. Location choice: each firm chooses its location $y_i \in [0, 1/2]$ on her spoke;

3. Price choice: given the location y_i, the firm chooses the price schedule $p_i(x_s)$.

The game is solved by backward induction to identify strategies which are undominated and constitute a sub-game perfect equilibrium. The following analysis closely parallels Lederer-Hurter[14], of which this game constitutes a generalization.

3.2 The Price Equilibrium

Suppose for now that firms have announced their price schedule, given the selected location y_i over their own spoke: $p_i(x_s|y_i)\ \forall i = 1...n$. Customers at location x_s choose to buy from the firm providing the good at the lowest price[7]. Having defined X as the set of all possible locations over all N spokes, the following partition of X from the point of view of firm i can be introduced:

$$D_i(p_i, p_{-i}) = \{x \in X \text{ s.t. } p_i(x|y_i) < \min\{p_{-i}(x|y_{-i}\}\}$$

$$D_S(p_i, p_{-i}) = \{x \in X \text{ s.t. } p_i(x|y_i) = \min\{p_{-i}(x|y_{-i}\}\}$$

The sets D_i and D_S can be interpreted as the segments of demand faced by the $i = 1...n$ firms respectively and in which $D_S(p_i, p_{-i})$ is the market region shared by two or more firms. To complete the definition of the firms' demand schedules a sharing rule is specified. This is needed to assign the contended region of the market D_S to a specific firm. Consistent with most of the existing literature, a cost-advantage (or *efficient*) sharing rule is adopted. The implications and the role of this assumption are discussed in presenting the results of the paper.

[7] When no ambiguity is possible, the notation x is used from now on instead of x_s.

Definition 1 *A **cost advantage sharing rule** is a function r such that:*

$$r(y_i, p_i, y_{-i}, p_{-i}, x) = \begin{cases} 0 & \text{if} \quad c_i + td(y_i, x) > \min\{c_{-i} + td(y_{-i}, x)\} \\ \\ r_i & \text{if} \quad c_i + td(y_i, x) = \min\{c_{-i} + td(y_{-i}, x)\} \\ \\ 1 & \text{if} \quad c_i + td(y_i, x) < \min\{c_{-i} + td(y_{-i}, x)\} \end{cases}$$

$r_i \in]0, 1[$ represents the random share in case parity persists, provided that $\sum_i r_i = 1$. On the basis of the specified sharing rule, the set D_S can be partitioned in the subsets $D_{S_i} = \{x \in D_S | r_i = 1\}$, for which clearly holds the following: $\bigcup_{\forall i} D_{S_i} = D_S$.
The profit function of firm i is then defined as:

Definition 2 *Given that customers are uniformly distributed over the N spokes and that there exist a sharing rule r, the profit function of firm i is:*

$$\pi_i^r(y_i, p_i, y_{-i}, p_{-i}, x) = \frac{2}{N} \int_{D_i} [p_i(x|y_i) - td(y_i, x) - c_i]\, dx$$
$$+\frac{2}{N} \int_{D_S} [p_i(x|y_i) - td(y_i, x) - c_i]\, r(y_i, p_i, y_{-i}, p_{-i}, x) dx$$

In order to characterize the equilibrium price schedule, it is useful to identify the boundaries which the firm faces in setting the price at a given location. The two following remarks allow to characterize these boundaries.

Remark 1 *Given the set of locations $y = (y_1, ..., y_i, ..., y_n)$ chosen by firms at the first stage, firm i can not make losses in serving customer x:*

$$p_i(x|y) \geq c_i + td(y_i, x) \quad \forall x \in X$$

If this was not the case, from Definition 2 is clear that the customer x would contribute negatively to profits: this is not rational for the firm.

Remark 2 *Given the set of locations* $y = (y_1, ..., y_i, ..., y_n)$ *chosen by firms at the first stage, firm* i *can not price the good delivered to customer* x *over the reservation value:*

$$p_i(x|y) \leq v \quad \forall x \in X$$

A delivered price above the customers' (known) reservation value would not only imply that the market may not be fully covered but it is also privately irrational, as it would drive down to zero the chances of firm i to serve customer x.
The following proposition characterizes the unique pure strategy equilibrium of the second subgame:

Proposition 1 *Given the set of locations* $y = (y_1, ..., y_i, ..., y_n)$*, the unique equilibrium of the price subgame is:*

$$p_i^*(x|y) = \max\{c_i + td(y_i, x), \min\{c_{-i} + td(y_{-i}, x)\}\} \qquad \forall i = 1...n \tag{1}$$

The proposition establishes that the equilibrium price schedule is closely linked to the cost structure. As a consequence of undercutting, the price at a generic location x is either the firm's cost of delivering the product or, if the firms is the lowest cost provider, the cost of the firm that is the second most efficient in delivering the good.

3.3 The Location Equilibrium

The equilibrium price schedule identified by (1) implies that the profit function for firm i can be written as:

$$\pi_i^r(y_i, p_i, y_{-i}, p_{-i}, x) = \frac{2}{N} \int_{D_i} \min\{c_{-i} + td(y_{-i}, x)\} - (c_i + td(y_i, x))dx$$

The Nash equilibrium of the location subgame is defined as:

$$y_i^* = \arg\max_{y_i \in X} \pi_i^r(y_i, p_i, y_{-i}^*, p_{-i}^*) \quad \forall i = 1...n$$

Before proceeding to the characterization of the location sub-game equilibrium, it is worth noticing that two possible cases can arise in the spokes model. The first possibility is that firms occupy all of the existing spokes, so that $n = N$: this is a n-firm generalization of the Hotelling[12]-Hoover[11] case. The second possible market structure is constituted by only n firms entering the market, made up by N spokes as $n < N$.

3.3.1 Location equilibrium in the $n = N$ case

In this section the existence and properties of equilibrium are analyzed in case the number of firms on the market equals the number of spokes. In order to characterize the equilibrium and its properties it is useful to define social cost as:

Definition 3 *The **social cost** is the total cost afforded by firms to supply the good to all customers on the market in a cooperative/cost minimizing way. Given a vector of locations $y = (y_1, ..., y_i, ...y_n)$, then:*

$$SC(y) = \frac{2}{N} \int_X \min_{\forall i} \{c_i + td(y_i, x)\} dx$$

It is important to notice that social cost is a continuous function of y over the support X. An important relation between profits and social cost exists and it is captured by the following expression:

$$\pi_i^r(y_i, p_i, y_{-i}, p_{-i}, x) = \frac{2}{N} \int_{D_i} \min\{c_{-i} + td(y_{-i}, x)\} - (c_i + td(y_i, x))dx =$$

$$= \frac{2}{N} \left[\int_X \min\{c_{-i} + td(y_{-i}, x)\}dx - \int_X \min_{\forall j} \{c_j + td(y_j, x)\}dx \right] =$$

$$= \frac{2}{N} \int_X \min\{c_{-i} + td(y_{-i}, x)\}dx - SC(y) \quad (2)$$

Although this is a case in which algebra is far more clear than words, an intuitive explanation of (2) is as follows: the profits of a firm consist of two elements. The first is positive and it is made in the region where the firm is the lowest cost provider, in other words on D_i. If the firm is the lowest cost provider on a region, she concurs to the definition of social cost in that region. On D_i the profits obtained are the differential between firm's delivery cost and the second most efficient firm's delivery cost, by (1). The other part is constituted by the rest of the market X on which the firm is not the lowest cost provider and, as such, does not contribute in computing the social cost but it does not make any profit either. Profits, then, are defined as the difference, on all the market X, between the lowest cost rival and the social cost, which in region D_i is just firm's cost while out of D_i is nothing but the lowest cost rival. Relation (2) allows to establish the following important results.

Proposition 2 *A* ***price-location equilibrium*** *exists in the spokes model with delivered products and* $n = N$*.*

Once established the existence of equilibrium, Proposition 3 outlines the properties of this case.

Proposition 3 *The vector* $y = (y_1^*, .., y_i^*, .., y_n^*)$ *is an equilibrium of the spokes model with delivered product when* $n = N$ ***if and only if:***

$$SC(y_i^*, y_{-i}^*) \leq SC(y_i, y_{-i}^*) \quad \forall y_i \in X \;\; \forall i = 1...n \tag{3}$$

and equilibrium price policies (1) are used by all firms.

The results provided establish that a price-location equilibrium exists in the spokes model when $n = N$. The most important feature is to show that the location chosen in order to maximize firms profits is also minimizing the sum of transportation costs, i.e. it is socially optimal. The result can be interpreted as follows: the competitive pressure between firms takes prices down to cost; given that a cost-advantage sharing rule is adopted, then profits are maximized at the location which is also minimizing the joint cost of serving the market. This result constitutes an extension of Lederer-Hurter[14], Theorem 3, to the case of n firms. Propositions 2 and 3 imply that their findings are robust in the spokes model: despite the gains from possible deviations are multiplied by $(n-1)$ in this setting, these are yet not profitable and an equilibrium exists. For completeness, it has to be stressed that this result is at the same time a special case of Theorem 3 in Lederer-Hurter[14] as the location space in the spokes model is one-dimensional: a corollary is that the equilibrium location vector y is also globally cost minimizing and, given the price schedule, it corresponds to the location profile chosen by a multi-plant monopolist.

Finally, notice the following corollary of the general result obtained.

Remark 3 *If all firms are symmetric, the competitive and socially optimal location it is just at the half of each firms' spoke, i.e.* $y^* = \hat{y} = \frac{1}{4}$.

As opposed to the next case $(n < N)$, in this setting symmetry between firms implies a symmetric outcome of the game.

3.3.2 Location equilibrium in the spokes model when $n < N$

In this section it is maintained that there is an exogenous number of firms n in the market. This number, however, is smaller than the number of spokes N. In such a setting the unique pure strategy equilibrium of the price sub-game is still described in Proposition 1. The intuition is as follows. A cost advantage sharing rule as in Definition 1 is adopted. For all occupied spokes, the lowest cost firm prices at the delivered cost of the second most efficient competitor. For non-occupied spokes, if there exist a firm with a cost advantage, she captures all the customers by pricing at the most efficient rival's delivered cost; if there is not a most efficient firm, all competitors price equally at the common delivered cost. The equilibrium price schedules are then still described by (1).

Turning to the location sub-game, the following strategy is used to prove the existence of the equilibrium and to characterize it. First, it is ruled out that the outcome of the game is symmetric: this is true in case all firms are *a priori* identical too. Second, an asymmetric equilibrium outcome is proved to exist and characterized. Finally, a mixed strategy equibrium of the location game is analyzed.

Non Existence of Symmetric Outcomes The main difference between this case and the previous one, in in which $n = N$, is the presence of empty spokes. The consumers on empty parts of the market do not have an *a priori* favourite firm and the remaining firms on the market start from even ground when trying to attract them to purchase their product. This feature affects the competitive forces in operation and has an impact on the outcome of the game.

Suppose that all firms produce with the same technology (i.e. $c_i = c$ $\forall i = 1..n$) : in other words all firms are symmetric in every respect. In such a situation, it is interesting to ask whether a symmetric equilibrium of the game exists.

Suppose all firms are perfectly symmetric and so are their strategy choices, the following non-existence result can then be established:

Proposition 4 *Assume that all firms are symmetric and the equilibrium price policies (1) are employed, then a **symmetric pure strategy equilibrium** of the location subgame **does not exist** in the spokes model with delivered product as $n < N$.*

Proposition 4 establishes the non-existence of a pure-strategy equilibrium in the location subgame spokes model with delivered product when $n < N$. The intuition for this result is the following: suppose first that the centre, where all the spokes join, is the symmetric equilibrium location of all firms. In that case, firms obtain no profit and they have a strictly positive unilateral incentive to deviate to a location internal to their own spokes. However, as the equilibrium location is a vector of points internal to the spokes, then it can be shown that a firm faces a unilateral incentive to move towards the centre to undercut all competitors and serve a larger share of the market. This implies that no equilibrium configuration exists when all firms are symmetric.

Equilibrium Outcomes Consider then asymmetric equilibrium outcomes. Before addressing the case of technologically symmetric firms, it is convenient to establish the existence and properties of the equilibrium in the more general case. Then, conditions under which a pure strategy price-location equilibrium with symmetric firms of the spokes model exists and $n < N$ are provided. It turns out that a small amount of asymmetry between firms is enough to guarantee the existence of an equilibrium. The next proposition also characterizes the equilibrium configuration.

Proposition 5 *Assume the equilibrium price policies (1) are employed. If $c_i < \min_{\forall j \neq i}\{c_j\}$, then a* ***pure strategy equilibrium*** *of the location subgame* ***exists*** *in the spokes model with delivered product as $n < N$. Moreover, the equilibrium location configuration is:*

$$y_i^* = \frac{1}{2} \qquad y_j^* = \frac{1}{6} + \frac{c_i - c_j}{3t} \qquad \forall j \neq i$$

Proposition 5 implies that in order to have a pure strategy equilibrium it is sufficient that there exist a lowest cost firm differing from all others, which can still be symmetric. The intuition for this result goes as follows. First, firms do not have an incentive to locate at the centre of the market, otherwise all but the lowest cost firm would get zero profits. They can instead get strictly positive profits by locating in the interior of their spoke. In that case, it can be verified that the lowest cost firm has an incentive to increase her location as much as possible. This tendency finds a limit at the centre, which is the location chosen by the most efficient firm. She targets all segments of the market which do not have a favourite brand and captures them all by locating at the middle. Incidentally, this implies also serving her own consumers and some consumers located on rivals' spokes. Other firms, instead, maximize their profits by "specializing", i.e. serving only part of their own spokes. It is also worth noting that the optimal location of a given firm is independent of the number of firms, the number of spokes and the costs of other less efficient firms. To conclude, a small amount of asymmetry allows firms to coordinate and a competitive equilibrium to exist: this can be interpreted as if the efficient firm is supplying a "general purpose" product while other firms focus on "niches" of the market.

Once determined the existence of the equilibrium and characterized the optimal locations in case firms are not perfectly symmetric, it is interesting to see how these compare with the social optimal location configuration.

Proposition 6 *Suppose* $c_i < \min_{\forall j \neq i}\{c_j\}$ *, then the vector of locations* $\hat{y}$ *minimizing the sum of the costs of delivery is given by:*

$$\hat{y}_i = \frac{1}{2} \qquad \hat{y}_j = \frac{1}{6} + \frac{c_i - c_j}{3t} \qquad \forall j \neq i$$

in the spokes model with delivery when $n < N$.

Social cost is minimized when the most efficient firm locates at the centre of the market. The social cost decreases as she increases her location but the centre where all spokes join provides her with a limit to that expansion. Cost minimization, then, clearly, implies all other firms to choose a location in the interior of their spokes. The last step is to provide the comparison of the socially optimal location with the competitive equilibrium in the asymmetric case. From Proposition 5 and 6 is clear that the optimal choice of both types of firms (lowest marginal cost and not) coincide with the locations that minimize social cost. The results of Lederer-Hurter[14] are robust to the existence of regions of the market where no firm locates, provided that the firms can choose an asymmetric locations' configuration. These results are summarized in the following proposition, stated without proof as it is straightforward:

Proposition 7 *In the spokes model with delivery if* $n < N$ *and* $c_i < \min_{\forall j \neq i}\{c_j\}$*, the degree of differentiation is socially optimal.*

After the more general case with asymmetric marginal costs, it is really straightforward to characterize the case in which all firms have access to the same technology. Despite the symmetry of cost, the locational outcome is asymmetric. The results are summarized in the following proposition:

Proposition 8 *Suppose* $c_i = c \;\; \forall i = 1...n$ *and assume the equilibrium price policies (1) are employed, then an asymmetric* ***pure strategy equilibrium*** *of the location subgame* ***exists*** *in the spokes model with delivered product as* $n < N$*. Moreover, in the spokes model with delivery when* $n < N$*, the equilibrium location configuration is given by the vector* $\hat{y}$*:*

$$\hat{y}_i = \frac{1}{2} \qquad \hat{y}_j = \frac{1}{6} \quad \forall j \neq i$$

The vector of locations $\hat{y}$ *coincides with the socially optimal locations' vector.*

The symmetric case is then simply a special case of the asymmetric one and shares most properties with it: in particular, the (asymmetric) equilibrium outcome is socially optimal. However, a further observation is in order. The existence of areas of the markets over which no firm has a cost advantage implies that the equilibrium location configuration is asymmetric even if all firms are perfectly identical. As mentioned above, this can be interpreted in terms of one firm supplying a "general purpose" product while all other specialize on their segment of the market. In this case, however, it is not *ex-ante* possible to predict which firm i will supply the generic product. There are in fact n possible equilibrium configurations. This implies that *a priori* it is not possible to forecast which parts of the market will experience specialization and which others will not and, from the point of view of a regulator, coordination-type of problems may arise.

An intuitive explanation of these results may be provided by MacLeod-Norman-Thisse[15]. In standard spatial models in which consumers travel to the outlet and buy the product, transportation costs can be thought as a measure of disutility and location is a product characteristic. In presence of product delivery, instead, the situation can be interpreted as firms trying to personalize and adapt their products to the demand expressed by consumers. The results presented can be seen as an explanation why some firms producing to order are specializing in a very specialized range of items while other supply a wider product line. A possible example is provided by taylor-made clothes or made to order shoes: traditional hand-crafting laboratories usually specialize on a very limited range of products. Large multinational companies, favoured by the recent developments in internet-based shopping, allow consumers to personalize their product by choosing between a wide number of characteristics and items.

Figure 2 illustrates the location equilibrium in case $n = 4$ firms are on a market composed of $N = 5$ spokes. In the same example, the equilibrium price schedule of firms serving a specific spoke is the following:

$$p_i\left(x_s, \tfrac{1}{6}\right) = \begin{cases} 0.2 + 0.8\left(\frac{1}{2} - x_s\right) & \text{if} \quad 0 < x_s \leq \frac{1}{3} \quad s = i \\ \\ 0.2 + 0.8\left(x_s - \frac{1}{6}\right) & \text{if} \quad \frac{1}{3} < x_s \leq \frac{1}{2} \quad s = i \\ \\ 0.2 + 0.8\left(\frac{5}{6} - x_s\right) & \text{if} \quad x_s \in X,\ x_s \notin X_i \end{cases}$$

while for the firm located at the centre is:

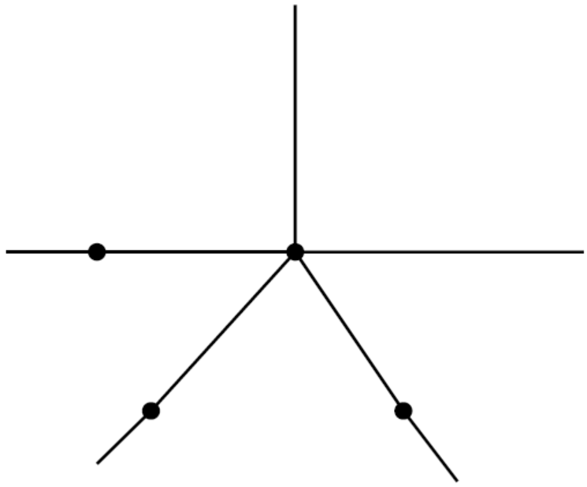

Figure 2: Equilibrium location configuration as $n < N$, with $n = 4$ symmetric firms and $N = 5$ spokes.

$$p_j\left(x_s, \tfrac{1}{2}\right) = \begin{cases} 0.2 + 0.8\left(\frac{5}{6} - x_s\right) & \text{if} \quad 0 < x_s \leq \frac{1}{2} \quad s = j \\ 0.2 + 0.8\left(x_f - \frac{1}{6}\right) & \text{if} \quad x_f \in X,\ x_f \notin X_j, \frac{1}{3} < x_f < \frac{1}{2} \\ 0.2 + 0.8\left(\frac{1}{2} - x_f\right) & \text{if} \quad x_f \in X,\ x_f \notin X_j, 0 \leq x_f \leq \frac{1}{3} \\ 0.2 + 0.8\left(\frac{5}{6} - x_e\right) & \text{if} \quad 0 \leq x_e < \frac{1}{2} \end{cases}$$

assuming $c = 0.2$ and $t = 0.8$.

3.3.3 Mixed Strategy Equilibria of the Location Game

The price-location game studied has n pure strategy equilibria and coordination failures may arise. In such a setting it is of interest to consider also the mixed strategy equilibria of the game. Mixed strategy equilibria have an intutive appeal: the probability distribution obtained can be interpreted as a prediction on the location pattern in a market with a geographical or characteristics structure as the spokes model. The question is whether this location pattern is socially optimal or not.

Before turning to the mixed strategies equilibrium, the socially optimal location configuration has to be considered. Consider symmetric firms: the social optimum is characterized then by the following Proposition:

Proposition 9 *Suppose all firms are symmetric. The vector of locations $\hat{y}$ that minimizes the sum of the costs of delivery is given by:*

$$\hat{y}_i = \frac{n^2 + N - n}{4n^2} \qquad \forall i = 1..n$$

in the spokes model with delivery when $n < N$.

It can be noticed that as the number of empty spokes $N - n$ increases, the social cost minimizing location shifts towards the centre of the market. In fact, when the all spokes are occupied and firms are all identical it is easily checked that the social minimizing location corresponds to $y_i = \frac{1}{4}$ for all firms. However, when there are empty spokes the numerator in the expression above increases and the optimal location is $y_i > \frac{1}{4}$. In case the number of empty spokes is particularly large (i.e. if $N - n \geq n^2$), then minimum differentiation $\hat{y} = \frac{1}{2}$ is the socially optimal choice.

Proposition 10 characterizes the mixed strategy equilibrium of the spokes model as $n < N$ in the simplest case: $n = 2$ and $N = 3$.

Proposition 10 *The equilibrium price policies (1) are employed. If $n = 2$ and $N = 3$ and both firms choose absolutely continuous distribution functions with a connected support, then the spokes model with delivered product has a **mixed strategy equilibrium** in the location subgame where all firms select*

$$f(y_i) = \frac{2}{\mathrm{erf}\{\frac{\sqrt{2}}{2}\}} \sqrt{\frac{2}{\pi}} e^{-2y_i^2} \qquad \forall y_i \in \left[0, \frac{1}{2}\right] \qquad \forall i = 1..n$$

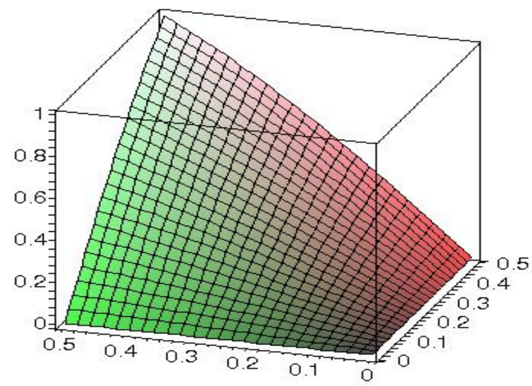

Figure 3: Joint Cumulative Distribution Function of the Mixed Strategy Equilibrium.

Table 1. Properties of the MSNE

	$F(y_i)$
Mean	0.2299
Median	0.2208
Variance	0.0199
Std Dev	0.1411

where erf(.) denotes the error function. Figure 3 plots the joint cumulative distribution function obtained. The characteristics of the distribution function characterizing the mixed strategy equilibrium are reported in Table 1. It can be noticed that both the expected and the median location imply a slightly larger degree of differentiation with respect to the case in which the number of spokes is identical to the number of firms participating in the market. More interestingly, however, the mean and the median locations can be compared with the socially optimal location. The following result can be stated[8]:

[8] This result is reported without proof as it follows directly from the expressions previously reported.

Proposition 11 *In the spokes model with $n = 2$ and $N = 3$, the expected location of the MSNE of the game ($E(y_i) = 0.2299$) implies a sub-optimal excess differentiation with respect to the social cost minimizing configuration ($\hat{y}_i = 0.3125$).*

The failure of firms to coordinate when competing for the empty spokes implies two types of inefficiencies in their location choices. First, by definition a mixed strategy equilibrium involves uncertainty on the location of the firms. In the mixed strategy equilibrium outlined for the special case considered, the volatility of the location choice, as measured by the standard deviation, is 0.14. A further source of inefficiency is linked to the discrepancy between social optimality and the expected value of location under the mixed strategy. The example considered seems to suggest that firms are expected to choose an amount of differentiation superior to what would be socially optimal. The interpretation for this result is that fierce competition for the empty spokes makes them less profitable in expectation; this is suggesting firms to be conservative and focus relatively more on their own market turf.

This result may have an interesting empirical implication and help explain the spread of firms around the center of a market. Given the strong assumptions adopted, this interpretation can not be overemphasized; however, the conclusion may provide a key to understand a pattern often observed in urban industrial development. Industrial firms usually tend to deliver their products to final retailers or consumers. In urban agglomerates groups of firms tend to cluster in districts closer to the periphery than the centre of the market.

4 Concluding Remarks

This paper establishes the existence and the properties of the equilibria of the spokes model when firms are in charge of delivering their product to customers and they can practice perfect price discrimination with respect to consumers location.

The main results of the paper are the following: Lederer-Hurter[14] provide an important existence result and the characterization of the equilibrium in a spatial duopoly with delivery. This paper generalizes their analysis to the spokes model. The robustness of their results is confirmed in a n firms spatial context which is a direct generalization of the Hotelling linear market. Competition drives prices down to delivered costs. Location choice corresponds to the social optimum, no matter if firms locate on all spokes or not. In case all spokes are occupied there is an obvious and intuitive relation between profit maximization and social cost minimization. This is driving the optimality result on location. In case there are empty spokes, instead, it is shown that the only possible equilibrium configuration involves asymmetric locations. If firms are not totally symmetric, the result can be interpreted as follows: firms optimally coordinate so that the most efficient provides a "general purpose" product, that can be differentiated for targeting the regions of the market which are not covered by rivals. All other firms concentrate on their own "niche". However, if all firms are symmetric, a coordination problem arises: it is not possible *a priori* to know which firm will serve a wider market by serving the generic product as n asymmetric equilibrium location configurations exist. Finally, an atomless mixed strategy with connected support is considered. The distribution function obtained as an equilibrium can be interpreted as a possible equilibrium location pattern. In that case, it is found that the average location displays a socially suboptimal excess of differentiation.

These results provide a contribution to two different streams of literature. First, the theory of horizontal product differentiation and, in particular, on the recently introduced spokes model. It is shown that if firms deliver their product a fully competitive equilibrium of the model exists. The result is in sharp contrast with the standard version of the spokes model, in which

firms charge a uniform price: in that setting, tractability requires consumers to have preferences for only two varieties of the product and not for all. Moreover, for the first time the location issue is addressed within the spokes model.
Second, the results shed new light on optimal firm location under delivered product. The existence of an equilibrium in the spokes model not only constitutes a generalization of the results of Lederer-Hurter[14] to the case of n firms. The case in which the number of firms is lower than the number of spokes displays interesting features. The distinguishing characteristic of the equilibrium when the number of firms is identical to the number of spokes is that location choice coincides in case firms are profit maximizers and in case they aim at social cost minimization. This important and desirable result holds also in case empty spokes exist. The resulting equilibrium location configuration, however, is now asymmetric: one firm supplies a "general purpose" product targeting the empty spokes while all other focus on their own "niches". The previous interpretation can be related to the literature on general purpose products. In contrast with the existing literature[9], the spokes model with product delivery generates "endogenously" an equilibrium configuration characterized by a general purpose product and niches.

Three extensions are worth exploring in the future. First, the results obtained imply that only one firm produces the general product. An interesting development would be to find conditions under which one or more firms opt for a general purpose product while other focus on targeted ones.

A second extension of this research is to find conditions under which a location equilibrium can be found in the spokes model with mill prcing. This case is absolutely technically challenging. However, it would be interesting to address the impact of non covered segments of the market on optimal

[9] The seminal paper on general purpose products is Von Ungern Sternberg[21]. Further results have been provided then by Hendel-Neiva de Figuereido[10] and Doraszelski-Draganska[6].

location of firms also in that case.

Finally, the theory developed delivers important empirical predictions. It is not difficult to find geographical markets in which firms deliver their product. More challenging is to find markets whose characteristics are fully captured by the spokes model. The model, however, can be adapted to test the predictions on of the theory on optimal location.

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A Appendix

The Appendix contains the proofs of all the propositions stated in the text apart from Proposition 7, 8 and 11.

A.1 Proof of Proposition 1

Remarks 1 and 2 have ruled out all possible prices **not** included in the following subset:

$$p_i(x|y) \in [\, c_i + td(y_i, x),\ v\,]$$

For a given y, having assumed a cost advantage sharing rule r firm i can match any offer of a rival firm j as long as she is the most efficient in serving customer x.

To begin with, consider firm i. First the claim for which, in equilibrium, the price $p_i^*(x|y)$ is identical for all firms $i = 1...n$. Having defined above: $D_{S_i} = \{x \in D_S | r_i = 1\}$ the subset of the market region D_S in which firm i has a cost advantage. Then, assuming *ad absurdum* that:

$$\forall x \in D_{S_i},\ p_i(x|y) - p_j(x|y) = \epsilon > 0,\ j \neq i$$

then firm i loses all the customers located in x. On the other hand, proceeding again *ad absurdum*:

$$\forall x \in D_{S_i},\ p_j(x|y) - p_i(x|y) = \epsilon > 0,\ j \neq i$$

then firm i can raise its price and increase the profit margin on customers located at x. Then, the only possibility left is that: $p_i(x|y) = p_j(x|y)$. The reasoning can be repeated for all $j \neq i$ and for all $i = 1...n$.

Second, $p_i^*(x|y) = \max\{c_i + td(y_i, x), \min\{c_{-i} + td(y_{-i}, x)\}\}$. Suppose instead that, for $x \in D_{S_i}$, the following holds:

$$p_i^*(x|y) - \max\{c_i + td(y_i, x), \min\{c_{-i} + td(y_{-i}, x)\}\} = \epsilon > 0$$

In that case, the second most efficient firm, say j, can choose the price $p_j(x|y) = p_i^* - \xi$ and for sufficiently small ξ raise its profit, which contradicts the definition of equilibrium. The reasoning can be repeated for all $j \neq i$, all $i = 1...n$.

Analogous reasoning allows to establish the result for the subsets of X over which $r_i = r$ and $r_i = 0$. *Q.E.D.*

A.2 Proof of Proposition 2

As the strategy space X_i is non empty, convex and compact, then it is sufficient to show that the profit function

$$\pi_i^r(y_i, p_i, y_{-i}, p_{-i}, x) = \frac{2}{N} \int_{D_i} \min\{c_{-i} + td(y_{-i}, x)\} - (c_i + td(y_i, x))dx$$

is continuous and quasi-concave in y_i so that the classical existence results by Debreu-Glicksberg-Fan[10] can be applied. This is done in what follows:

- **Continuity in** (y_i, y_{-i}) The expression of profits in this case is:

  $$\pi_i^r(y_i, p_i, y_{-i}, p_{-i}, x) = \frac{2}{N} \int_{D_i} \min_{\forall j \neq i}\{c_j + td(y_j, x)\} - (c_i + td(y_i, x))dx$$

  This can be rearranged as follows:

  $$\pi_i^r(y_i, p_i, y_{-i}, p_{-i}, x) = \frac{2}{N}\left[\min_{\forall j \neq i}\left\{c_j[x]_{D_i} + t\int_{D_i} d(y_j, x)dx\right\}\right.$$
  $$\left. - \left(c_i[x]_{D_i} + t\int_{D_i} d(y_i, x)dx\right)\right]$$

  Given the definition of distance adopted in Section 2, the integrals can be written as:

  $$\int_{D_i} d(y_i, x)dx = \int_0^{y_i} d(y_i, x)dx + \int_{y_i}^{x^*} d(y_i, x)dx$$

[10] The result is reported in Dasgupta-Maskin[5], Proposition 1.

$$\int_{D_i} d(y_j, x)dx = \int_0^{x^*} d(y_j, x)dx$$

if $x^* < \frac{1}{2}$ and:

$$\int_{D_i} d(y_i, x)dx = \int_0^{y_i} d(y_i, x)dx + \int_{y_i}^{1/2} d(y_i, x)dx + \int_{1-x^*}^{1/2} d(y_i, x)dx$$

$$\int_{D_i} d(y_j, x)dx = \int_0^{1/2} d(y_j, x)dx + \int_{1-x^*}^{1/2} d(y_j, x)dx$$

if $x^* > \frac{1}{2}$ in which

$$x^* = \frac{c_j - c_i}{2t} + \frac{1}{2}(1 - y_j + y_i)$$

is the indifferent consumers between firm i and j, the lowest cost firm between the rivals. All the functions involved are continuous. All the transformations required to compute the profit functions (integration, multiplication, addition and subtraction) are preserving continuity.
The profit function is:

$$\pi_i^r(y_i, y_{-i}) = \frac{2}{N}\left\{\int_0^{x^*} \min_{\forall j \neq i}\{c_j + t(1 - x - y_j)\}dx\right.$$
$$\left. - \left[\int_0^{y_i} c_i + t(y_i - x)dx + \int_{y_i}^{x^*} c_i + t(x - y_i)dx\right]\right\}$$

in case $x^* < 1/2$ and:

$$\pi_i^r(y_i, y_{-i}) = \frac{2}{N}\left\{\int_0^{x^*} \min_{\forall j \neq i}\{c_j + t(1 - x - y_j)\}dx\right.$$
$$- \left[\int_0^{y_i} c_i + t(y_i - x)dx + \int_{y_i}^{x^*} c_i + t(x - y_i)dx\right]$$
$$\left. +(N-1)\left[\int_{1-x^*}^{1/2} \min_{\forall j \neq i}\{c_j + t(1 - x - y_j\}dx - \int_{1-x^*}^{1/2} c_i + t(1 - x - y_i)dx\right]\right\}$$

in case $x^* > 1/2$.
As all the functions involved in the computation of the profit function

in the two different scenarios are continuous, then the only possible discontinuity may take place when y_i and y_j are such that a shift takes place from $x^* < 1/2$ to $x^* > 1/2$. Define:

$$\overline{y}_i : \{y_i \in [0, 1/2] | \, x^*(y_i, y_j) = 1/2\}$$

and

$$\overline{y}_{-i} : \{y_j \in [0, 1/2] | \, x^*(y_i, y_j) = 1/2\}$$

In order to verify wether this is the case, consider the following limits:

$$LD_i = \lim_{y_i \to \overline{y}_i^-} \pi_i(y_i, y_{-i}) = \lim_{y_i \to \overline{y}_i^+} \pi_i(y_i, y_{-i}) = LU_i =$$
$$\frac{c_j t^2 - c_i t^2 - 4c_j t + 4c_i t + 2t^2 - 8c_j^2 + 16c_j c_i + 16c_j y_j t - 8c_i^2 - 16c_i y_j t - 8y_j^2 t^2}{4Nt^2}$$

As the limits are identical for all values of y_j, then it can be concluded that the profit function is continuous for all values of y_i. A similar reasoning can be applied to verify continuity with respect to y_{-i}, which in fact requires continuity only in y_j, the location of the lowest cost rival. Computing the following limits:

$$LD_{-i} = lim_{y_{-i} \to \overline{y}_{-i}^-} \pi_i(y_i, y_{-i}) = \frac{c_j t - c_i t + 2t - 4c_j + 4c_i - 8y_i^2 t}{4Nt}$$

$$LU_{-i} = lim_{y_{-i} \to \overline{y}_{-i}^+} \pi_i(y_i, y_{-i}) = \frac{c_j t - c_i t + 2t - 4c_j + 4c_i - 8y_i^2 t}{4Nt}$$

it is easy to verify they are identical. This allows to conclude the profit function is also continuous in y_{-i}. The reasoning can then be repeated for the profit functions of all other $n-1$ firms, obtaining an identical result. This is implying that the profit function is continuous with respect to both y_i and y_j for $y_i \in [0, 1/2]$ $y_j \in [0, 1/2]$ in all the possible cases: $x^* < \frac{1}{2}$, $x^* > \frac{1}{2}$ and $x^* = \frac{1}{2}$ and $\forall j \neq i$.

- **Quasi-Concavity in y_i**

Using expression (2), the profits can be written as:

$$\pi_i^r(y_i, p_i, y_{-i}, p_{-i}, x) = \frac{2}{N}\int_{D_i} \min\{c_{-i} + td(y_{-i}, x)\} - (c_i + td(y_i, x))dx =$$

$$\frac{2}{N}\left[\int_X \min\{c_{-i} + td(y_{-i}, x)\}dx - \int_X \min_{\forall j}\{c_j + td(y_j, x)\}dx\right] =$$

$$\frac{2}{N}\int_X \min\{c_{-i} + td(y_{-i}, x)\}dx - SC(y)$$

All the term in expression but social cost are independent of y_i. In order to show that the profit function is quasi-concave in y_i it is sufficient to show that the social cost function is quasi-convex in y_i. The social cost as a function of y_i can be written as follows:

$$SC(y_i) = c_i[x]_{D_i} + t\int_{D_i} d(y_i, x)dx$$

As all the functions are continuous, quasi-convexity can be checked by computing the second derivative and verifying it has a positive sign. Again three cases need to be considered. The second derivative of the social cost function is:

$$\frac{\partial^2 SC(y_i)}{\partial y_i^2} = \begin{cases} \frac{5t}{2N} & \text{if } \ x^* < 1/2 \\ \frac{4t}{N} & \text{if } \ x^* = 1/2 \\ \frac{9t}{2N} & \text{if } \ x^* > 1/2 \end{cases}$$

which is clearly positive in all cases, then social cost is quasi-convex. This proves the quasi-concavity of π_i with respect to y_i.

These results, in conjunction with Proposition 1, prove the claim: a price location equilibrium of the spokes model exists as $n = N$. *Q.E.D.*

A.3 Proof of Proposition 3

If $y = (y_1^*, .., y_i^*, .., y_n^*)$ is a vector of equilibrium locations, then:

$$\pi_i^r(y_i^*, p_i^*, y_{-i}^*, p_{-i}^*) \geq \pi_i^r(y_i, p_i^*, y_{-i}^*, p_{-i}^*) \quad \forall y_i \in X \ \ \forall i = 1...n$$

which, by (2), can be written as:

$$\frac{2}{N}\int_X \min\{c_{-i}+td(y_{-i},x)\}dx - SC(y_i^*,y_{-i}^*) \geq \frac{2}{N}\int_X \min\{c_{-i}+td(y_{-i},x)\}dx - SC(y_i,y_{-i}^*)$$

from which (3) follows immediately.

If instead $y = (y_1^*, .., y_i^*, .., y_n^*)$ satisfies (3) and the equilibrium price schedule (1) is employed by firms, then by (2) and Proposition 1 $y = (y_1^*, .., y_i^*, .., y_n^*)$ is a price-location equilibrium of the spokes model when $n = N$. *Q.E.D.*

A.4 Proof of Proposition 4

Suppose first that the vector of equilibrium locations is $y^* = (\frac{1}{2}, ..., \frac{1}{2})$, i.e. the centre of the market. In this case all firms obtain zero profits, as no one has cost advantage in delivering the product:

$$c + td(x, y_i^*) = \min\{c + td(x, y_{-i}^*)\} \quad \forall\ x \in X$$

which is implying that:

$$p_i(x|y) = c + td(x, y_i^*) \quad \forall\ x \in X$$

so that $\pi_i^r = 0\ \forall i = 1..n$. However, this implies that each firm has a private unilateral incentive to deviate from $y_i^* = \frac{1}{2}$ and choose a location internal to her own spoke $y_i \in [0, \frac{1}{2}[$. If the deviation is $\delta > 0$ towards the interior of the spokes, then:

$$c + td(x, y_i^* - \delta) < \min_{\forall j \neq i}\{c + td(x, y_j^*)\} \quad \forall\ x \in D_i$$

where D_i, the market served by firm i, is now constituted by consumers on her own spoke with a location such that i faces the lowest cost in delivering to them, i.e. $D_i = \{x \in X_i | x \in [0, \frac{1}{2} - \frac{\delta}{2}]\}$. This implies that firm i makes a positive mark-up on the market served and has a strictly positive profit:

$$\pi_i^r = \int\limits_{D_i} \min_{\forall j \neq i}\{c + td(x, y_j^*)\} - [c + td(x, y_i^* - \delta)]dx > 0$$

This proves that firms have a unilateral incentive to deviate, contradicting the definition of a pure strategy Nash equilibrium. Then $y^* = (\frac{1}{2}, ..., \frac{1}{2})$ can not be an equilibrium.

Suppose, then, the equilibrium vector y^* is such that $y_i^* \in [0, \frac{1}{2}[\ \forall i = 1..n$. The profits received by firms are:

$$\pi_i^r(y^*) = \int\limits_0^{\frac{1}{2}} \min_{\forall j \neq i}\{c + td(x, y_j^*)\} - [c + td(x, y_i^*)]dx$$

If vector y^* were to be the equilibrium, firms should not have an incentive to deviate. However, suppose firm i moves in the direction of the centre of the market by $\delta > 0$. In that case the profits of firm i are:

$$\pi_i^r(y_i^* + \delta, y_{-i}^*) = \int\limits_{D_i} \min_{\forall j \neq i}\{c + td(x, y_j^*)\} - [c + td(x, y_i^* + \delta)]dx$$

which can be re-expressed as:

$$\pi_i^r(y_i^* + \delta, y_{-i}^*) = \int\limits_0^{\frac{1}{2}} \min_{\forall j \neq i}\{c + td(x, y_j^*)\} - [c + td(x, y_i^* + \delta)]dx +$$

$$+(N - n)\int\limits_0^{\frac{1}{2}} \min_{\forall j \neq i}\{c + td(x, y_j^*)\} - [c + td(x, y_i^* + \delta)]dx +$$

$$+(n - 1)\int\limits_{\frac{1}{2} - \frac{\delta}{2}}^{\frac{1}{2}} \min_{\forall j \neq i}\{c + td(x, y_j^*)\} - [c + td(x, y_i^* + \delta)]dx$$

It is possible then to compute the profit differential as:

$$\Delta\pi_i^r(y^*,\delta) = \pi_i^r(y_i^*+\delta, y_{-i}^*) - \pi_i^r(y^*)$$

which can be explicitly computed:

$$\Delta\pi_i^r(y^*,\delta) = -\underbrace{\int_0^{\frac{1}{2}}[c+td(x,y_i^*+\delta)] - [c+td(x,y_i^*)]dx}+$$

$$\underbrace{+(N-n)\int_0^{\frac{1}{2}}\min_{\forall j\neq i}\{c+td(x,y_j^*)\} - [c+td(x,y_i^*+\delta)]dx}+$$

$$\underbrace{+(n-1)\int_{\frac{1}{2}-\frac{\delta}{2}}^{\frac{1}{2}}\min_{\forall j\neq i}\{c+td(x,y_j^*)\} - [c+td(x,y_i^*+\delta)]dx}$$

Substituting the correct expressions for the distance and after the appropriate algebraic manipulations it is found:

$$\Delta\pi_i^r(y^*,\delta) = \pi_i^r(y_i^*+\delta, y_{-i}^*) - \pi_i^r(y^*) = \frac{1}{4}t\delta[8y_i^* - 2 + 2(N-n) + \delta(n-1)]$$

As $N-n>0$ and $n-1\geq 0$ it follows that $\Delta\pi_i^r(y^*,\delta)>0$ $\forall y^*\in[0,\frac{1}{2}[$. This implies for all possible symmetric equilibrium configuration firms have an unilateral incentive to deviate. This contradicts the definition of pure strategy Nash equilibrium. *Q.E.D.*

A.5 Proof of Proposition 5

Suppose first that the equilibrium configuration is $y_i = y_j = \frac{1}{2}$ for all firms $j\neq i$ where i represents the lowest marginal cost firm. In that case, the profits obtained by firms are:

$$\pi_j^r\left(\frac{1}{2}\right) = 0 \quad \forall j\neq i$$

$$\begin{aligned}
\pi_i^r\left(\frac{1}{2}\right) &= \frac{2}{N}\int\limits_X \min_{\forall i\neq j}\{c_j + td(y_j,x)\}dx - \frac{2}{N}\int\limits_X [c_j + d(y_i,x)]dx = \\
&= \frac{2}{N}\int\limits_X (\min_{\forall i\neq j}\{c_j\} - c_i)dx + \frac{2t}{N}\int\limits_X [d(y_j,x) - d(y_i,x)]dx = \\
&= \min_{\forall i\neq j}\{c_j\} - c_i
\end{aligned}$$

i.e. all but the most efficient firm get zero profits. This implies that, provided that the most efficient firm chooses $y_i = \frac{1}{2}$, all other firm choose a location belonging to the interior of their spoke. This implies the game reduces to an Arrow-Enthoven problem. This can be written as:

$$\max_{y_j} \pi_j^r(y_j,y_i) = \frac{2}{N}\int\limits_0^{x_{ij}} [c_i + td(y_i,x)] - [c_j + d(y_j,x)]dx \qquad y_i, y_j \in \left[0, \frac{1}{2}\right]$$

$$\begin{aligned}
\max_{y_i} \pi_i^r(y_i,y_j) = \frac{2}{N}\int\limits_0^{\frac{1}{2}} \min_{\forall k\neq i}\{c_k + td(y_k,x)\} - [c_i + td(y_i,x)]dx + \\
+\frac{2}{N}\sum_{k\neq i}\int\limits_{x_{ik}}^{\frac{1}{2}} [c_k + d(y_k,x)] - [c_i + td(y_i,x)]dx + \\
+\frac{2}{N}(N-n)\int\limits_0^{\frac{1}{2}} \min_{\forall k\neq i}\{c_k + td(y_k,x)\} - [c_i + td(y_i,x)]dx
\end{aligned}$$

where:

$$x_{ik} = \frac{c_j - c_i}{2t} + \frac{1 - y_i + y_j}{2}$$

represents the consumer on j-th spoke which is indifferent between firm j and firm i. Were the maximization unconstrained, firm i had an incentive to choose a location $y_i > \frac{1}{2}$:

$$\left.\frac{\partial \pi_i^r(y_i,y_j)}{\partial y_i}\right|_{y_i=\frac{1}{2}} = \frac{c_j - c_i}{t} + \frac{2N-(n+3)}{2(n-1)} - y_j > 0$$

implying, that under our assumptions on $N > n$, $c_j > c_i$ and $y_j \in [0, \frac{1}{2}]$, the optimal choice for the most efficient firm must be $y_i^* = \frac{1}{2}$. This implies that the problem for firm j has an internal solution given by:

$$y_j^* = \frac{1}{6} + \frac{1}{3t}(c_i - c_j) \quad \forall j \neq i$$

Q.E.D.

A.6 Proof of Proposition 6

By Definition 3, social cost can be written as:

$$SC(y_1 ... y_n) = \frac{2}{N} \int\limits_X \min_{\forall i} \{c + td(x, y_i)\} dx$$

which more explicitly is:

$$SC(y_i, y_j) = \frac{2}{N} \sum_{k \neq i} \int\limits_0^{x_{ik}} [c_k + t|y_k - x|] dx + \frac{2}{N} \int\limits_0^{\frac{1}{2}} [c_i + t|y_i - x|] dx +$$

$$+ \frac{2}{N}(N - n) \int\limits_0^{\frac{1}{2}} [c_i + t(1 - y_i - x)] dx + \frac{2}{N} \sum_{k \neq i} \int\limits_{x_{ik}}^{\frac{1}{2}} [c_i + t(1 - y_i - x)] dx$$

The problem reduces to:

$$\min_{y_i, y_j} SC(y_i, y_j)$$
$$\text{s.t. } y_i, y_j \in \left[0, \frac{1}{2}\right]$$

The unconstrained maximization would suggest that the most efficient firm i should choose location $y_i > \frac{1}{2}$. This can be shown by verifying that, under the assumptions made on $n < N$, c_i and c_j:

$$\left. \frac{\partial SC(y_i, y_j)}{\partial y_i} \right|_{y_i = \frac{1}{2}} = -t - \frac{n(c_j + c_i)}{2N} + \frac{3n + 1}{4N} t + \frac{(nty_j + 4)}{2N} < 0$$

holding for all possible $y_j \in [0, \frac{1}{2}]$. If the constraint for the location of firm i is binding, then the social cost minimization problem implies an internal solution for all other $n-1$ less efficient firms. In particular, solving the problem:

$$\min_{y_j} SC\left(\frac{1}{2}, y_j\right) \qquad \forall j \neq i$$

leads to the following conclusion:

$$\hat{y}_i = \frac{1}{2} \qquad \hat{y}_j = \frac{1}{6} + \frac{1}{3t}(c_i - c_j) \qquad \forall j \neq i$$

Q.E.D.

A.7 Proof of Proposition 9

By Definition 3, social cost can be written as:

$$SC(y_1 ... y_n) = \frac{2}{N}\left[\sum_{j=1}^{n} \int_{S_j} \min_{\forall j}\{c + td(x, y_j)\} dx + \frac{N-n}{n} \int_{S_l} c + td(x, y_l) dx\right]$$

which more explicitly is:

$$SC(y_1 ... y_n) = \frac{2}{N}\left[n \int_0^{\frac{1}{2}} c + t|y_i - x| dx + \frac{N-n}{n} \int_0^{\frac{1}{2}} c + t(1 - y_i - x) dx\right]$$

from which is easily found that:

$$\hat{y}_i = \arg\min_{y_i}\{SC(y_1 ... y_n)\} = \frac{n^2 + N - n}{4n^2} \qquad \forall i = 1..n$$

Q.E.D.

A.8 Proof of Proposition 10

Suppose that the opponent firm j has chosen the strategy $f(y_j)$. At any mixed strategy, firm i must have the same expected payoff defined by:

$$\int_{\alpha}^{y_i} \pi_i^r(y_i > y_j) f(y_j) dy_j + \int_{y_i}^{\beta} \pi_i^r(y_i > y_j) f(y_j) dy_j = C$$

where C is a given fixed constant and:

$$\begin{aligned} \pi_i^r(y_i \quad > \quad y_j) = & \int_0^{\frac{1}{2}} [c + td(x, y_j) - c + td(x, y_i) dx] + \\ & + \int_0^{\frac{1}{2}} [c + td(x, y_j) - c + td(x, y_i)] dx + \\ & + \int_{x_j^*}^{\frac{1}{2}} [c + td(x, y_j) - c + td(x, y_i)] dx \end{aligned}$$

while:

$$\pi_i^r(y_i < y_j) = \int_0^{x_{ij}^*} [c + td(x, y_j) - c + td(x, y_i)] dx$$

After computing the expressions for the profits in the two cases, the condition on the expected payoff can be written as:

$$\begin{aligned} & \int_{\alpha}^{y_i} \frac{t}{2N} [-3y_i^2 + y_j^2 - 2y_j y_i + 6y_i - 6y_j + 2] f(y_j) dy_j + \\ + & \int_{y_i}^{\beta} \frac{t}{2N} [-3y_i^2 + y_j^2 - 2y_i y_j + 2y_i - 2y_j + 1] f(y_j) dy_j = C \end{aligned}$$

Differentiating with respect to y_i yields:

$$(-4y_i^2+2)f(y_i)+\int_{\alpha}^{y_i}(-6y_j+6-2y_j)f(y_j)dy_j+$$
$$-(1-4y_i^2)f(y_i)-\int_{y_i}^{\beta}(2-6y_i-2y_j)f(y_j)dy_j=0$$

Observe that β can only be equal to $\frac{1}{2}$. Suppose $\beta<\frac{1}{2}$, this would imply that firm i choose the centre of the market with probability zero: in that case the opponent firm would have an incentive to locate exactly at the centre and earn a strictly positive profits. This leads to:

$$f(y_i)+6(1-y_i)F(y_i)-2F(y_i)(1-y_i)-1-(2-6y_i)F(y_i)+1+2(y_i-1)F(y_i)=0$$

The mixed strategy satisfies the following first order differential equation:

$$f(y_i)=-4y_iF(y_i)$$

whose solution is:

$$\frac{dF(y_i)}{dy_i}=ke^{-2y_i^2}$$

where the constant k is chosen in a way that:

$$\int_{\alpha}^{\frac{1}{2}}\frac{dF(y_i)}{dy_i}dy_i=1$$

and:

$$\int_{0}^{\alpha}\frac{dF(y_i)}{dy_i}dy_i=0$$

It can be shown that the only α compatible with such a requirement are $\alpha=0$ and so the distribution function is:

$$f(y_i)=\frac{2}{\mathrm{erf}\{\frac{\sqrt{2}}{2}\}}\sqrt{\frac{2}{\pi}}e^{-2y_i^2}\quad\forall y_i\in\left[0,\frac{1}{2}\right]\quad\forall i=1..n$$

Q.E.D.