Supplementary Appendix to the Article Entitled

Specification Testing in Nonstationary Time Series Models

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This supplementary document provides the detailed proofs of the main results in Sections 2 and 3 as well as the technical lemmas in Appendix A of the main article.

Appendix B: Detailed proofs of the main results

We next give the detailed proofs of the main results in Sections 2 and 3 of the main article. In the sequel, we use C to denote a positive constant whose value may vary from place to place, and for brevity of analysis, we assume, without loss of generality, that the dimensionality of Θ is $d = 1$.

Proof of Theorem 2.1(i). Note that under the sequence of local alternative hypotheses H_1^L, we have

$$
\begin{aligned}
Q_n(h) &= \sum_{t=1}^n \sum_{s=1,\neq t}^n \widehat{e}_t K_{s,t} \widehat{e}_s \\
&= \sum_{t=1}^n \sum_{s=1,\neq t}^n e_t K_{s,t} e_s + \sum_{t=1}^n \sum_{s=1,\neq t}^n \widetilde{g}_t K_{s,t} \widetilde{g}_s + \sum_{t=1}^n \sum_{s=1,\neq t}^n \Delta_n(V_t) K_{s,t} \Delta_n(V_s) + \\
&\quad 2\sum_{t=1}^n \sum_{s=1,\neq t}^n \Delta_n(V_t) K_{s,t} \widetilde{g}_s + 2\sum_{t=1}^n \sum_{s=1,\neq t}^n \widetilde{g}_t K_{s,t} e_s + 2\sum_{t=1}^n \sum_{s=1,\neq t}^n \Delta_n(V_t) K_{s,t} e_s \\
&\equiv \sum_{i=1}^6 \overline{Q}_{n,i}(h), \qquad \text{(B.1)}
\end{aligned}
$$

where $\widetilde{g}_t = g(V_t, \theta_0) - g(V_t, \widehat{\theta})$.

Using Lemma A.4(i) and (ii) and arguments similar to those in the proof of Theorem 2.1 in Gao *et al* (2009b), we have, as $n \to \infty$,

$$
\frac{\overline{Q}_{n,1}(h)}{\widetilde{\sigma}_{n,1}} = \frac{\overline{Q}_{n,1}(h)}{\sigma_{n,1}} + o_P(1) \to_D \mathsf{N}(0,1) \quad \text{and} \quad \frac{\overline{\sigma}_n^2 - \widetilde{\sigma}_{n,1}^2}{\widetilde{\sigma}_{n,1}^2} = o_P(1), \qquad \text{(B.2)}
$$

where $\sigma_{n,1}^2 = 2\sigma_e^4 \sum_{t=1}^{n} \sum_{s=1,\neq t} K_{s,t}^2$, $\overline{\sigma}_n^2 = 2\sum_{t=1}^n \sum_{s=1,\neq t}^n \widehat{e}_t^2 K^2\left(\frac{V_t - V_s}{h}\right)\widehat{e}_s^2$, $\widetilde{\sigma}_{n,1}^2 = 2\widetilde{\sigma}_e^4 \sum_{t=1}^{n} \sum_{s=1,\neq t}^{n} K_{s,t}^2$ and $\widetilde{\sigma}_e^2 = \frac{1}{n}\sum_{t=1}^{n} e_t^2$.

Lemma A.4(i) and (B.2) imply

$$c_1^* n\sqrt{n}h(1 + o_P(1)) \leq \overline{\sigma}_n^2 \leq c_2^* n\sqrt{n}h(1 + o_P(1)) \tag{B.3}$$

for some constants $0 < c_1^* < c_2^* < \infty$. This result will be frequently used when we evaluate the orders of $\overline{Q}_{n,i}(h)$ for $i = 2, \cdots, 6$ in the subsequent proof.

We first show that

$$\frac{\overline{Q}_{n,2}(h)}{\overline{\sigma}_n} = o_P(1). \tag{B.4}$$

In order to prove (B.4), we will need to deal with the following term:

$$(\widehat{\theta} - \theta_0)^2 \sum_{t=1}^{n} \sum_{s=1,\neq t}^{n} \dot{g}_\theta(V_t, \theta_0) K_{s,t} \dot{g}_\theta(V_s, \theta_0), \tag{B.5}$$

where $\dot{g}_\theta(v, \theta)$ is the first derivative of $g(v, \theta)$ with respect to θ.

Recalling that $p_t(\cdot)$ is the marginal density function of V_t and letting $q_t(\cdot)$ be the marginal density of $\frac{V_t}{\sqrt{t}}$, by Lemma A.1 in the main article, there exists a positive constant c_0 such that

$$p_t(v) = \frac{1}{\sqrt{t}} q_t\left(\frac{v}{\sqrt{t}}\right) \leq \frac{c_0}{\sqrt{t}} \tag{B.6}$$

uniformly for v as $t \to \infty$. Let $q_{st}(\cdot, \cdot)$ be the joint density function of $\left(\frac{V_t - V_s}{\sqrt{t-s}}, \frac{V_s}{\sqrt{s}}\right)$ and $q_{st}(\cdot|u)$ be the conditional density of $\frac{V_t - V_s}{\sqrt{t-s}}$ given $\frac{V_s}{\sqrt{s}} = u$ for $t > s$. By Lemma A.1, we can show that as $t - s \to \infty$,

$$p_{st}(v, u) = \frac{1}{\sqrt{(t-s)s}} q_{st}\left(\frac{v-u}{\sqrt{t-s}}, \frac{u}{\sqrt{s}}\right) = \frac{1}{\sqrt{(t-s)s}} q_{st}\left(\frac{v-u}{\sqrt{t-s}}\Big|\frac{u}{\sqrt{s}}\right) q_s\left(\frac{u}{\sqrt{s}}\right) \leq \frac{c_0}{\sqrt{(t-s)s}}, \tag{B.7}$$

where $p_{st}(\cdot, \cdot)$ is the joint density function of (V_t, V_s).

Without loss of generality, we assume that both (B.6) and (B.7) hold for all $t > s$. We are now ready to evaluate the order of (B.5). For any small enough $\zeta > 0$ and a large enough $c(\zeta)$ with $c(\zeta) > \left(\int_{-\infty}^{\infty} \dot{g}_\theta^2(v, \theta_0) dv / \zeta\right)^2$, using (B.6) and (B.7), we have

$$\begin{aligned}
& \mathsf{P}\Big(\Big|\sum_{t=1}^{n} \sum_{s=1, s\neq t}^{n} \dot{g}_\theta(V_t, \theta_0) K_{s,t} \dot{g}_\theta(V_s, \theta_0)\Big| > c(\zeta) nh\Big) \\
\leq\ & \frac{1}{nhc(\zeta)} \sum_{t=1}^{n} \sum_{s=1,\neq t}^{n} \mathsf{E}\Big[\Big|\dot{g}_\theta(V_t, \theta_0) K\left(\frac{V_s - V_t}{h}\right) \dot{g}_\theta(V_s, \theta_0)\Big|\Big] \\
\leq\ & \frac{[c(\zeta)]^{1/3}}{nhc(\zeta)} \sum_{t=2}^{n} \sum_{s=1}^{t-1} \frac{h}{\sqrt{(t-s)s}} \int_{-\infty}^{\infty} \dot{g}_\theta^2(v, \theta_0) dv \\
\leq\ & \frac{\int_{-\infty}^{\infty} \dot{g}_\theta^2(v, \theta_0) dv}{[c(\zeta)]^{1/2}} < \zeta,
\end{aligned}$$

which implies

$$\sum_{t=1}^{n}\sum_{s=1,s\neq t}^{n}\dot{g}_\theta(V_t,\theta_0)K_{s,t}\dot{g}_\theta(V_s,\theta_0)=O_P\left(nh\right). \tag{B.8}$$

By Lemma A.2(i), we have, as $n\to\infty$

$$\begin{aligned}&\left(\widehat{\theta}-\theta_0\right)^2\sum_{t=1}^{n}\dot{g}_\theta(V_t,\theta_0)K_{st}\dot{g}_\theta(V_s,\theta_0)\\ =\ & O_P\big((\delta_n^2+n^{-1/2})nh\big)=O_P(\delta_n^2nh)+O_P\big(\sqrt{n}h\big)\\ =\ & o_P\big(n^{3/4}\sqrt{h}\big)=o_P(\overline{\sigma}_n),\end{aligned} \tag{B.9}$$

when $\delta_n=o\left(n^{-1/8}h^{-1/4}\right)$. We thus complete the proof of (B.4).

We next evaluate the orders of $\overline{Q}_{n,i}(h)$, $i=3,\ 4,\ 5,\ 6$. By Definition 2.5, we can find an integrable function $\overline{\Gamma}(\cdot)$ such that

$$\max\left\{\Delta_n^2(v),|\Delta_n(v)\dot{\Delta}_n(v)|\right\}\leq\delta_n^2\overline{\Gamma}(v). \tag{B.10}$$

Using (B.3) and (B.10) and a derivation similar to that of (B.8), we have, as $n\to\infty$

$$\overline{Q}_{n,3}(h)=O_P\left(\delta_n^2nh\right)=o_P\big(n^{3/4}\sqrt{h}\big)=o_P(\overline{\sigma}_n) \tag{B.11}$$

as $\delta_n=o\left(n^{-1/8}h^{-1/4}\right)$.

For $\overline{Q}_{n,4}(h)$, note that

$$\begin{aligned}\overline{Q}_{n,4}^2(h)\ &=\ 4\Big[\sum_{t=1}^{n}\sum_{s=1,\neq t}^{n}\widetilde{g}_tK_{s,t}^{1/2}K_{s,t}^{1/2}\Delta_n(V_s)\Big]^2\\ &\leq\ 4\Big|\sum_{t=1}^{n}\sum_{s=1,\neq t}^{n}\widetilde{g}_tK_{s,t}\widetilde{g}_s\Big|\cdot\Big|\sum_{t=1}^{n}\sum_{s=1,\neq t}^{n}\Delta_n(V_t)K_{s,t}\Delta_n(V_s)\Big|\\ &\leq\ C|Q_{n,2}(h)|\cdot|Q_{n,3}(h)|,\end{aligned} \tag{B.12}$$

which, together with (B.4) and (B.11), leads to

$$\frac{\overline{Q}_{n,4}(h)}{\overline{\sigma}_n}=o_P(1). \tag{B.13}$$

In order to deal with $\overline{Q}_{n,5}(h)$, we first evaluate the second moment of the following form

$$\overline{Q}_{n,51}(h)\equiv\sum_{t=2}^{n}\Big[\sum_{s=1}^{t-1}K\big(\frac{V_t-V_s}{h}\big)\dot{g}_\theta(V_s,\theta_0)\Big]e_t. \tag{B.14}$$

In order to simplify the notation, define $A(v)=\dot{g}_\theta(v,\theta_0)$. Note that

$$\begin{aligned}&\sum_{t=2}^{n}\mathsf{E}\Big[\sum_{s=1}^{t-1}K\big(\frac{V_t-V_s}{h}\big)A(V_s)\Big]^2=\sum_{t=2}^{n}\sum_{s=1}^{t-1}\mathsf{E}\Big[K\big(\frac{V_t-V_s}{h}\big)A(V_s)\Big]^2\\ &+2\sum_{t=3}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}\mathsf{E}\Big[K\big(\frac{V_t-V_{s_1}}{h}\big)K\big(\frac{V_t-V_{s_2}}{h}\big)A(V_{s_1})A(V_{s_2})\Big]\\ &\equiv\ \overline{Q}_{n,51}(1)+\overline{Q}_{n,51}(2).\end{aligned} \tag{B.15}$$

We only consider the case of $1 \le s_2 < s_1 < t \le n$ as other cases can be handled analogously. For ease of exposition, let $V_{s_1,t} = V_t - V_{s_1}$ and $V_{s_2,s_1} = V_{s_1} - V_{s_2}$. Using the same arguments as those in the derivations of Lemma A.1 in Gao *et al* (2009b) and (B.8), we have

$$\begin{aligned}
&\overline{Q}_{n,51}(2) = 2\sum_{t=3}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1} \mathsf{E}\Big[K\big(\frac{V_t - V_{s_1}}{h}\big)K\big(\frac{V_t - V_{s_2}}{h}\big)A(V_{s_1})A(V_{s_2})\Big] \\
= &\ 2\sum_{t=3}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1} \int\int\int K\big(\frac{y_1}{h}\big)K\big(\frac{y_1+y_2}{h}\big)A(y_2+y_3)A(y_3)p(y_1,y_2,y_3)dy_3dy_2dy_1 \\
= &\ 2\sum_{t=3}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1} \frac{1}{\sqrt{s_2}}\frac{1}{\sqrt{s_1-s_2}}\frac{1}{\sqrt{t-s_1}} \int\int\int K\big(\frac{y_1}{h}\big)K\big(\frac{y_1+y_2}{h}\big)A(y_2+y_3)A(y_3) \\
&\times q\left(\frac{y_1}{\sqrt{t-s_1}}, \frac{y_2}{\sqrt{s_1-s_2}}, \frac{y_3}{\sqrt{s_2}}\right) dy_3dy_2dy_1 \\
= &\ O\big(n^{\frac{3}{2}}h^2\big), \qquad \text{(B.16)}
\end{aligned}$$

where $p(\cdot,\cdot,\cdot)$ and $q(\cdot,\cdot,\cdot)$ are the joint density functions of

$$\big(V_{s_1,t}, V_{s_2,s_1}, V_{s_2}\big) \quad \text{and} \quad \big(\frac{1}{\sqrt{t-s_1}}V_{s_1,t}, \frac{1}{\sqrt{s_1-s_2}}V_{s_2,s_1}, \frac{1}{\sqrt{s_2}}V_{s_2}\big),$$

respectively. Similarly, we can show that

$$\overline{Q}_{n,51}(1) = O\left(nh\right) = o(n^{\frac{3}{2}}h^2). \qquad \text{(B.17)}$$

Thus, using Assumptions 1 and 2 as well as equations (B.15)–(B.17), we have, as $n \to \infty$,

$$\begin{aligned}
&\mathsf{E}\big[\overline{Q}_{n,51}(h)\big]^2 = \mathsf{E}\Big\{\sum_{t=2}^{n}\Big[\sum_{s=1}^{t-1}K\big(\frac{V_t - V_s}{h}\big)A(V_s)\Big]e_t\Big\}^2 = \sum_{t=2}^{n}\mathsf{E}\Big\{\Big[\sum_{s=1}^{t-1}K\big(\frac{V_t - V_s}{h}\big)A(V_s)\Big]^2 e_t^2\Big\} \\
&+2\sum_{t_1=3}^{n}\sum_{t_2=2}^{t_1-1}\sum_{s_1=1}^{t_1-1}\sum_{s_2=1}^{t_2-1}\mathsf{E}\Big[K\big(\frac{V_{t_1} - V_{s_1}}{h}\big)A(V_{s_1})K\big(\frac{V_{t_2} - V_{s_2}}{h}\big)A(V_{s_2})e_{t_2}e_{t_1}\Big] \\
&= \sum_{t=2}^{n}\mathsf{E}\Big\{\Big[\sum_{s=1}^{t-1}K\big(\frac{V_t - V_s}{h}\big)A(V_s)\Big]^2\mathsf{E}\big[e_t^2|\mathcal{F}_{t-1}\big]\Big\} + 2\sum_{t_1=3}^{n}\sum_{t_2=2}^{t_1-1}\sum_{s_1=1}^{t_1-1}\sum_{s_2=1}^{t_2-1} \\
&\mathsf{E}\Big\{\Big[K\big(\frac{V_{t_1} - V_{s_1}}{h}\big)A(V_{s_1})K\big(\frac{V_{t_2} - V_{s_2}}{h}\big)A(V_{s_2})e_{t_2}\Big]\mathsf{E}\big[e_{t_1}|\mathcal{F}_{t_1-1}\big]\Big\} \\
&= O\big(n^{\frac{3}{2}}h^2\big),
\end{aligned}$$

where we have used the conditions in Assumption 1(ii), i.e., $\mathsf{E}\left[e_t|\mathcal{F}_{t-1}\right] = 0$ and $\mathsf{E}\left[e_t^2|\mathcal{F}_{t-1}\right] = \sigma_e^2$ a.s. We then have

$$\overline{Q}_{n,51}(h) = O_P\big(n^{\frac{3}{4}}h\big). \qquad \text{(B.18)}$$

In order to deal with $\overline{Q}_{n,5}(h)$, in addition to $\overline{Q}_{n,51}(h)$, we also need to investigate the following term

$$\overline{Q}_{n,52}(h) \equiv \sum_{t=2}^{n} \Big[\sum_{s=1}^{t-1} K\big(\frac{V_t - V_s}{h}\big) e_s \Big] A(V_t), \tag{B.19}$$

where $A(v) = \dot{g}_\theta(v, \theta_0)$ is defined as above. A Taylor expansion of $A(\cdot)$ implies that

$$A(V_t) = A(V_s) + \dot{A}(V_s)(V_t - V_s) + C_{ts}(V_t - V_s)^2, \tag{B.20}$$

where $\dot{A}(v) = \frac{dA(v)}{dv}$ and C_{ts} is bounded by Assumption 3(ii) and the second I-regularity condition in Definition 2.3. Thus, in order to deal with (B.19), we need only to evaluate the orders of the following terms:

$$\begin{aligned}
\overline{Q}_{n,52}(1) &= \mathsf{E}\Big\{ \sum_{t=2}^{n} \Big[\sum_{s=1}^{t-1} K\big(\frac{V_t - V_s}{h}\big) A(V_s) e_s \Big] \Big\}^2, \\
\overline{Q}_{n,52}(2) &= h^2 \mathsf{E}\Big\{ \sum_{t=2}^{n} \Big[\sum_{s=1}^{t-1} \big(\frac{V_t - V_s}{h}\big) K\big(\frac{V_t - V_s}{h}\big) \dot{A}(V_s) e_s \Big] \Big\}^2, \\
\overline{Q}_{n,52}(3) &= h^4 \mathsf{E}\Big\{ \sum_{t=2}^{n} \Big[\sum_{s=1}^{t-1} C_{ts} \big(\frac{V_t - V_s}{h}\big)^2 K\big(\frac{V_t - V_s}{h}\big) e_s \Big] \Big\}^2.
\end{aligned}$$

Since $\overline{Q}_{n,52}(2)$ and $\overline{Q}_{n,52}(3)$ are of asymptotic orders smaller than that of $\overline{Q}_{n,52}(1)$, we next deal with $\overline{Q}_{n,52}(1)$ only. Observe that

$$\begin{aligned}
\mathsf{E}\Big\{ \sum_{t=2}^{n} \Big[\sum_{s=1}^{t-1} K\big(\frac{V_t - V_s}{h}\big) A(V_s) e_s \Big] \Big\}^2 &= \sum_{t=2}^{n} \mathsf{E}\Big\{ \Big[\sum_{s=1}^{t-1} K\big(\frac{V_t - V_s}{h}\big) A(V_s) e_s \Big] \Big\}^2 \\
+2 \sum_{t_1=3}^{n} \sum_{t_2=2}^{t_1-1} \sum_{s_1=1}^{t_1-1} \sum_{s_2=1}^{t_2-1} \mathsf{E}\Big[A(V_{s_1}) K\big(\frac{V_{t_1} - V_{s_1}}{h}\big) A(V_{s_2}) K\big(\frac{V_{t_2} - V_{s_2}}{h}\big) e_{s_1} e_{s_2} \Big]. &
\end{aligned} \tag{B.21}$$

Note that $V_{s,t} \equiv V_t - V_s = \sum_{i=s+1}^{t} v_i \overset{P}{\sim} \sum_{j=0}^{\infty} \phi_j \left(\sum_{i=s+1}^{t} \varepsilon_{i-j} \right)$ as $t - s \to \infty$. By Assumption 1(ii), we have, for all $1 < s < t$,

$$\begin{aligned}
& \mathsf{E}\left[e_s | \sigma(e_{s-1}, \cdots, e_1; \varepsilon_s, \cdots, \varepsilon_{-\infty}; \varepsilon_t, \cdots, \varepsilon_{s+1}) \right] \\
= \; & \mathsf{E}\left[e_s | (e_{s-1}, \cdots, e_1; \varepsilon_s, \cdots, \varepsilon_{-\infty}) \right] = 0 \quad a.s.
\end{aligned} \tag{B.22}$$

Thus, by equation (B.22), we have, for all $1 < s < t$

$$\mathsf{E}\left[e_s | \sigma\left(e_{s-1}, \cdots, e_1; v_t, \cdots, v_1 \right) \right] = 0 \quad a.s. \tag{B.23}$$

And similarly, we have, for all $1 < s < t$

$$\mathsf{E}\left[\left(e_s^2 - \sigma_e^2 \right) | \sigma\left(e_{s-1}, \cdots, e_1; v_t, \cdots, v_1 \right) \right] = 0 \quad a.s. \tag{B.24}$$

Using equations (B.23) and (B.24), we have

$$\sum_{t=2}^{n} \mathsf{E}\Big[\sum_{s=1}^{t-1} K\big(\frac{V_{s,t}}{h}\big)A(V_s)e_s\Big]^2 = \sum_{t=2}^{n}\sum_{s=1}^{t-1}\sigma_e^2 \cdot \mathsf{E}\Big[K^2\big(\frac{V_{s,t}}{h}\big)A^2(V_s)\Big] = O\left(nh\right). \quad \text{(B.25)}$$

In order to deal with the second term on the right hand side of equation (B.21), we consider the case where $1 \le s_2 \ne s_1 < t_2 < t_1 \le n$. By equation (B.22) and some basic calculations of conditional expectation, we have, for $1 \le s_2 \ne s_1 < t_2 < t_1$,

$$\mathsf{E}\Big[A(V_{s_1})K\big(\frac{V_{t_1}-V_{s_1}}{h}\big)A(V_{s_2})K\big(\frac{V_{t_2}-V_{s_2}}{h}\big)e_{s_1}e_{s_2}\Big] = 0. \quad \text{(B.26)}$$

Consider the case where $s_1 = s_2$ and $1 \le s_2 = s_1 < t_2 < t_1$. Similarly to the derivation in (B.25), it can be shown that, as $n \to \infty$,

$$\sum_{t_1=3}^{n}\sum_{t_2=2}^{t_1-1}\sum_{s_1=1}^{t_2-1} \mathsf{E}\Big[A^2(V_{s_1})K\big(\frac{V_{s_1,t_1}}{h}\big)K\big(\frac{V_{s_1,t_2}}{h}\big)e_{s_1}^2\Big] = O\big(n^{\frac{3}{2}}h^2\big). \quad \text{(B.27)}$$

Analogously to (B.26)–(B.27), one may also deal with other cases of s_1, s_2, t_1, and t_2, and the asymptotic order would not exceed $O\big(n^{\frac{3}{2}}h^2\big)$ as shown in (B.27). Therefore, by (B.21), (B.25)–(B.27), we have, as $n \to \infty$,

$$\overline{Q}_{n,52}(1) = \mathsf{E}\Big[\sum_{t=2}^{n}\sum_{s=1}^{t-1} K\big(\frac{V_{s,t}}{h}\big)A(V_s)e_s\Big]^2 = O\big(n^{\frac{3}{2}}h^2\big) + O\left(nh\right), \quad \text{(B.28)}$$

which implies that

$$\overline{Q}_{n,52}(h) = O_P\big(n^{\frac{3}{4}}h\big) + O_P\big(\sqrt{nh}\big). \quad \text{(B.29)}$$

Therefore, using Lemma A.2(i), (B.18) and (B.29), we can prove that

$$\begin{aligned}\overline{Q}_{n,5}(h) &= O_P\Big(\big(\widehat{\theta}-\theta_0\big)\sum_{t=1}^{n}\sum_{s=1,\ne t}^{n}\dot{g}_\theta(V_t,\theta_0)K_{s,t}e_s\Big) \\ &= O_P\big(\delta_n + n^{-\frac{1}{4}}\big)\cdot O_P\Big(n^{\frac{3}{4}}h + \sqrt{nh}\Big) = o_P(\overline{\sigma}_n) \quad \text{(B.30)}\end{aligned}$$

when $\delta_n = o\left(n^{-1/8}h^{-1/4}\right)$.

In a similar way, we may show that

$$\overline{Q}_{n,6}(h) = o_P(\overline{\sigma}_n). \quad \text{(B.31)}$$

The proof of Theorem 2.1(i) is completed by (B.1), (B.2), (B.4), (B.11), (B.13), (B.30) and (B.31). ■

Proof of Theorem 2.1(ii). Note that

$$\begin{aligned}\overline{Q}_{n,3}(h) &= \sum_{t=1}^{n}\sum_{s=1,\ne t}^{n}\Delta_n^2(V_s)K_{s,t} + \sum_{t=1}^{n}\sum_{s=1,\ne t}^{n}\big[\Delta_n(V_t)-\Delta_n(V_s)\big]K_{s,t}\Delta_n(V_s) \\ &\equiv \overline{Q}_{n,31}(h) + \overline{Q}_{n,32}(h). \quad \text{(B.32)}\end{aligned}$$

It is easy to see that $\overline{Q}_{n,31}(h)$ is the leading term of $\overline{Q}_{n,3}(h)$. Following the proof of Theorem 2.1(i), we need only to show that when $\delta_n n^{1/8}h^{1/4} \to \infty$,

$$\frac{1}{n^{3/4}h^{1/2}}\overline{Q}_{n,31}(h) \to_P \infty. \tag{B.33}$$

Recall that $\widehat{p}_n(V_t) = \frac{\sigma_\phi}{\sqrt{nh}}\sum_{s=1}^n K\big(\frac{V_s-V_t}{h}\big)$ and $V_{s,t} = \sum_{i=s+1}^t v_i$. Since the kernel function $K(\cdot)$ is symmetric and positive, we have, uniformly in $1 \le t \le \lfloor \frac{n}{2} \rfloor$,

$$\begin{aligned}
\widehat{p}_n(V_t) &= \frac{\sigma_\phi}{\sqrt{nh}}\sum_{s=1}^{t-1} K\big(\frac{V_{s,t}}{h}\big) + \frac{\sigma_\phi}{\sqrt{nh}}\sum_{s=t+1}^{n} K\big(\frac{V_{t,s}}{h}\big) + \frac{\sigma_\phi}{\sqrt{nh}}K(0) \\
&\ge \frac{\sigma_\phi}{\sqrt{nh}}\sum_{s=t+1}^{n} K\big(\frac{V_{t,s}}{h}\big) = \frac{\sqrt{n-t}}{\sqrt{n}} \cdot \frac{\sigma_\phi}{\sqrt{n-t}h}\sum_{s=1}^{n-t} K\big(\frac{\widetilde{V}_s(t)}{h}\big) \\
&= \frac{\sqrt{n-t}}{\sqrt{n}}\widetilde{p}_{(n-t)}(0) + o_P(1) = \frac{\sqrt{n-t}}{\sqrt{n}}L_B(1,0) + o_P(1),
\end{aligned} \tag{B.34}$$

where $\widetilde{V}_s(t) = \sum_{i=t+1}^{t+s} v_i$ and $\widetilde{p}_{(n-t)}(0) = \frac{\sigma_\phi}{\sqrt{n-t}h}\sum_{s=1}^{n-t} K\big(\frac{\widetilde{V}_s(t)}{h}\big)$. We next show that the last equality in (B.34) holds uniformly in $1 \le t \le \lfloor \frac{n}{2} \rfloor$. Defining $v_0^\diamond = 0$, $v_1^\diamond = v_{t+1}, \cdots, v_{n-t}^\diamond = v_n$, it is easy to see that $\widetilde{V}_s(t) = V_s^\diamond \equiv \sum_{t=1}^s v_t^\diamond$ and $V_s^\diamond$ can be regarded as a unit root process which satisfies the conditions in Lemma A.3. Note that

$$\lim_{n\to\infty}\widetilde{p}_{(n-t)}(0) = \lim_{m\to\infty}\frac{\sigma_\phi}{\sqrt{mh}}\sum_{s=1}^m K\big(\frac{V_s^\diamond - 0}{h}\big)$$

and the right hand side of the above equation is independent of t. Then, using (A.6) from Lemma A.3, we may prove that the last equality in (B.34) holds uniformly in $1 \le t \le \lfloor \frac{n}{2} \rfloor$. Equation (B.34) implies that

$$\begin{aligned}
\frac{\sigma_\phi}{\sqrt{nh}}\sum_{t=1}^n \Delta_n^2(V_t)\sum_{s=1,\ne t}^n K_{s,t} &\ge \frac{\sigma_\phi}{\sqrt{nh}}\sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \Delta_n^2(V_t)\sum_{s=1,\ne t}^n K_{s,t} \\
&= \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \Delta_n^2(V_t)\left(\widehat{p}_n(V_t) - \frac{\sigma_\phi}{\sqrt{nh}}K(0)\right) \\
&\ge \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \Delta_n^2(V_t)\Big(\sqrt{1-\frac{t}{n}}\,L_B(1,0) + o_P(1)\Big) \\
&\ge \frac{1}{\sqrt{2}}\sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \Delta_n^2(V_t)\,\big(L_B(1,0) + o_P(1)\big).
\end{aligned} \tag{B.35}$$

By Definition 2.5, there exists an integrable function $\widetilde{\Gamma}(\cdot)$ such that

$$\Delta_n^2(v) \ge \delta_n^2\widetilde{\Gamma}(v) \;\text{ and }\; \int_{-\infty}^{\infty}\widetilde{\Gamma}(v)dv > 0. \tag{B.36}$$

Then, by (B.35), (B.36) and Theorem 5.1 in Park and Phillips (1999), we have

$$
\begin{aligned}
\frac{\sigma_\phi}{nh\delta_n^2}\overline{Q}_{n,31}(h) &= \frac{\sigma_\phi}{nh\delta_n^2}\sum_{s=1}^{n}\Delta_n^2(V_s)\sum_{t=1,\neq s}^{n}K_{s,t} \\
&= \frac{1}{\sqrt{n}\delta_n^2}\sum_{s=1}^{n}\Delta_n^2(V_s)\left(\frac{\sigma_\phi}{\sqrt{n}h}\sum_{t=1}^{n}K_{s,t}-\frac{\sigma_\phi K(0)}{\sqrt{n}h}\right) \\
&\geq \frac{L_B(1,0)}{\sqrt{2n}}\sum_{s=1}^{\lfloor\frac{n}{2}\rfloor}\widetilde{\Gamma}(V_s)\left(1+o_P(1)\right) \\
&\to_P \frac{1}{2}\int_{-\infty}^{\infty}\widetilde{\Gamma}(v)dv\ L_B^2(1,0) \qquad \text{(B.37)}
\end{aligned}
$$

as $n\to\infty$. We can show that (B.33) holds if $\delta_n n^{1/8}h^{1/4}\to\infty$. Note that when $\delta_n n^{1/8}h^{1/4}\to\infty$, we can show that $\overline{Q}_{n,3}(h)$ is the leading term of $Q_n(h)$ asymptotically under H_1^L. The proof of Theorem 2.1(ii) is therefore completed. ■

Proof of Theorem 2.2(i). By Definition 2.6, there exists an asymptotically homogeneous function $\overline{\Lambda}(\cdot)$ such that

$$\max\left\{\Delta_n^2(v),\left|\Delta_n(v)\dot{\Delta}_n(v)\right|\right\}\leq\delta_n^2\overline{\Lambda}(v), \qquad \text{(B.38)}$$

where $\overline{\Lambda}(\lambda x)=v^2(\lambda)\overline{H}(x)+\overline{R}(x,\lambda)$, in which $\overline{H}(\cdot)$ is locally integrable and $\overline{R}(\cdot,\cdot)$ satisfies condition (i) or (ii) in Definition 2.6.

By (B.38), Lemma A.3 and Theorem 5.3 in Park and Phillips (1999), we have, as $n\to\infty$,

$$\overline{Q}_{n,3}(h) = \sum_{s=1}^{n}\sum_{t=1,\neq s}^{n}\Delta_n(V_s)K_{s,t}\Delta_n(V_t)=O_P\big(n^{3/2}v^2(\sqrt{n})h\delta_n^2\big),$$

which implies that (B.11) holds when $n^{3/8}\nu(\sqrt{n})h^{1/4}\delta_n\to 0$ holds. The rest of the proof is the same as that of Theorem 2.1(i). Details are thus omitted here. ■

Proof of Theorem 2.2(ii). By Definition 2.6, there exists an asymptotically homogeneous function $\widetilde{\Lambda}(\cdot)$ such that $\Delta_n^2(v)\geq\delta_n^2\widetilde{\Lambda}(v)$,

$$\widetilde{\Lambda}(\lambda x)=v^2(\lambda)\widetilde{H}(x)+\widetilde{R}(x,\lambda)\ \text{ and }\ \int_{-\infty}^{\infty}\widetilde{H}(v)L_B(1,v)dv>0. \qquad \text{(B.39)}$$

Similarly to the derivations in (B.35) and (B.37) and by $n^{3/8}\nu(\sqrt{n})h^{1/4}\delta_n\to\infty$, (B.39), Lemma A.3 and Theorem 5.3 in Park and Phillips (1999), we have

$$
\begin{aligned}
\frac{\sigma_\phi}{\sqrt{n}nL_B(1,0)v^2(\sqrt{n})h\delta_n^2}\overline{Q}_{n,31}(h) &= \frac{\sigma_\phi}{\sqrt{n}nL_B(1,0)v^2(\sqrt{n})h\delta_n^2}\sum_{s=1}^{n}\Delta_n^2(V_s)\sum_{t=1,\neq s}^{n}K_{s,t} \\
&\geq \frac{1}{nv^2(\sqrt{n})}\sum_{s=1}^{n}\widetilde{\Lambda}(V_s)\left(1+o_P(1)\right) \\
&\to_P \int_{-\infty}^{\infty}\widetilde{H}(v)\ L_B(1,v)dv \qquad \text{(B.40)}
\end{aligned}
$$

as $n \to \infty$, which implies that equation (B.33) holds. The proof of Theorem 2.2(ii) has therefore been completed. ■

Proof of Proposition 3.1. Recall that

$$\overline{Q}_{n,1}(h) = \sum_{t=1}^{n} \sum_{s=1,\neq t}^{n} e_t K_{s,t} e_s = \sum_{t=1}^{n} \sum_{s=1,\neq t}^{n} e_t K\big(\frac{V_s - V_t}{h}\big) e_s.$$

We next use Lemma C.3, which is given in Appendix C of this supplementary document, to prove (3.2) in Proposition 3.1. Let

$$a_{s,t} = \begin{cases} K_{s,t}/\sigma_{n,1}, & s \neq t, \\ 0, & s = t, \end{cases}$$

where $K_{s,t} = K_{t,s}$ due to the symmetry of $K(\cdot)$. Define $A_0^*(h)$ as $A_0(h)$ given in (3.4) with $K_{s,t}$ being replaced by $a_{s,t}$. Letting $\mathbf{A}_0$, $\mathbf{A}$, V, κ and $\mathbf{d}$ be defined as in the notations above Lemma C.3 in Appendix C, we can derive their explicit forms in our setting:

$$\mathbf{A}_0 = \mathbf{A} = A_0^*(h), \quad V = 0, \quad \mathbf{d} = \mathbf{0}, \quad \kappa = \rho_n(h), \tag{B.41}$$

where $\rho_n(h)$ is defined in (3.3). To prove (3.2), we only need to derive the rate for M_n which is defined in Lemma C.3 in Appendix C.

Note that, by Lemma A.4(i), we have

$$c_1 n^{3/2} h\ (1 + o_P(1)) \leq \sum_{t=1}^{n} \sum_{s=1,\neq t}^{n} K^2\big(\frac{V_t - V_s}{h}\big) \leq c_2 n^{3/2} h\ (1 + o_P(1)). \tag{B.42}$$

We next show that

$$\max_{1\leq t\leq n} \sum_{s=1,\neq t}^{n} K\big(\frac{V_t - V_s}{h}\big) \leq C_3 \sqrt{n} h\ (1 + o_P(1)) \tag{B.43}$$

for some constant $0 < C_3 < \infty$. By a standard argument, we have

$$\begin{aligned} \max_{1\leq t\leq n} \left\{ \frac{1}{\sqrt{n}h} \sum_{s=1,\neq t}^{n} K\left(\tfrac{V_t - V_s}{h}\right) \right\} &= \max_{1\leq t\leq n} \left\{ \frac{1}{\sqrt{n}h} \sum_{s=1}^{n} K\left(\tfrac{V_t - V_s}{h}\right) - \frac{K(0)}{\sqrt{n}h} \right\} \\ = \max_{1\leq t\leq n} &\left\{ \frac{1}{\sqrt{n}h} \sum_{s=1}^{n} K\left(\tfrac{V_t - V_s}{h}\right) \right\} + o_P(1). \end{aligned} \tag{B.44}$$

By Lemma C.4 in Appendix C, we have, for sufficiently large $C_3 > C_2^* > 0$,

$$\mathsf{P}\Big(\Big\{ \max_{1\leq t\leq n} \Big| \frac{1}{\sqrt{n}h} \sum_{s=1}^{n} K\big(\frac{V_t - V_s}{h}\big) \Big| > C_3 \Big\} \cap \Big\{ \max_{1\leq t\leq n} |V_t| \leq C_2 \sqrt{n} \Big\}\Big) = o(1). \tag{B.45}$$

Meanwhile, for any given small $\delta > 0$, by the functional limit theorem and continuous mapping, we have

$$\mathsf{P}\Big(\max_{1\le t\le n} |V_t| > C_2\sqrt{n}\Big) < \delta \tag{B.46}$$

for sufficiently large C_2. Equations (B.44)–(B.46) imply that

$$\begin{aligned}
&\mathsf{P}\Big(\max_{1\le t\le n}\Big|\frac{1}{\sqrt{n}h}\sum_{s=1}^{n} K\big(\frac{V_t - V_s}{h}\big)\Big| > C_3\Big)\\
= \ &\mathsf{P}\Big(\Big\{\max_{1\le t\le n}\Big|\frac{1}{\sqrt{n}h}\sum_{s=1}^{n} K\big(\frac{V_t - V_s}{h}\big)\Big| > C_3\Big\}\cap\Big\{\max_{1\le t\le n}|V_t| \le C_2\sqrt{n}\Big\}\Big) +\\
&\mathsf{P}\Big(\Big\{\max_{1\le t\le n}\Big|\frac{1}{\sqrt{n}h}\sum_{s=1}^{n} K\big(\frac{V_t - V_s}{h}\big)\Big| > C_3\Big\}\cap\Big\{\max_{1\le t\le n}|V_t| > C_2\sqrt{n}\Big\}\Big)\\
\le \ &\mathsf{P}\Big(\Big\{\max_{1\le t\le n}\Big|\frac{1}{\sqrt{n}h}\sum_{s=1}^{n} K\big(\frac{V_t - V_s}{h}\big)\Big| > C_3\Big\}\cap\Big\{\max_{1\le t\le n}|V_t| \le C_2\sqrt{n}\Big\}\Big) +\\
&\mathsf{P}\Big(\max_{1\le t\le n}|V_t| > C_2\sqrt{n}\Big) = o(1).
\end{aligned}\tag{B.47}$$

This completes the proof of (B.43).

Let $\lambda_{\max}$ be the maximal eigenvalue of the matrix $A_0^*(h)$ in absolute value and $\mathcal{L}_t$ be defined in the same way as in the notation above Lemma C.3. By (B.42) and (B.43), we have, as $n \to \infty$,

$$\lambda_{\max} \le \max_{1\le t\le n}\sum_{s=1,\ne t}^{n} a_{s,t} = O_P\Big(\frac{\sqrt{n}h}{\sqrt{n^{3/2}h}}\Big) = O_P\big(n^{-1/4}h^{1/2}\big), \tag{B.48}$$

and for some constant $\lambda_0 > 0$,

$$\|A_0^*(h)\| \equiv \sum_{t=1}^{n}\sum_{s=1,\ne t}^{n} a_{s,t}^2 \ge \lambda_0. \tag{B.49}$$

In view of (B.48) and (B.49), we have

$$\frac{\lambda_{\max}^2}{\|A_0^*(h)\|^2} = O_P\left(n^{-1/2}h\right). \tag{B.50}$$

Meanwhile, we have

$$\begin{aligned}
\sum_{t=1}^{n}\mathcal{L}_t^4 &= \sum_{t=1}^{n}\sum_{s=1,\ne t}^{n} a_{s,t}^4 + \sum_{s=1}^{n}\sum_{t_1=1}^{n}\sum_{t_2=1,\ne t_1}^{n} a_{s,t_1}^2 a_{s,t_2}^2\\
&= O_P\left(\sqrt{n}nh/(\sqrt{n}nh)^2 + n^2h^2/(\sqrt{n}nh)^2\right) = O_P(n^{-1}),
\end{aligned}\tag{B.51}$$

which implies

$$\frac{\big(\sum_{t=1}^{n}\mathcal{L}_t^4\big)^{1/2}}{\|A_0^*(h)\|} = O_P(n^{-1/2}). \tag{B.52}$$

With (B.50) and (B.52), we can show that the order of M_n (defined in Lemma C.3) is $O_P(n^{-1/2})$, which, together with (B.41) and Lemma C.3, proves (3.2) in Proposition 3.1.

We next prove (3.5) by using (3.2). Observe that under the null hypothesis H_0

$$Q_n(h) = \sum_{t=1}^{n} \sum_{s=1,\neq t}^{n} \widehat{e}_t K_{s,t} \widehat{e}_s = \overline{Q}_{n,1}(h) + \overline{Q}_{n,2}(h) + \overline{Q}_{n,5}(h), \tag{B.53}$$

where $\overline{Q}_{n,j}(h)$, $j = 1, 2, 5$, are defined in (2.3) of the main article. Define the event that

$$\mathcal{D}_n = \Big\{\Big|\frac{\overline{Q}_{n,2}(h) + \overline{Q}_{n,5}(h)}{\overline{\sigma}_n}\Big| > s_n\Big\} \cup \Big\{\Big|\frac{\sigma_{n,1} - \overline{\sigma}_n}{\sigma_{n,1}}\Big| > s_n\Big\},$$

where $s_n = n^{-1/8}h^{1/4}$, and let $\mathcal{D}_n^c$ be the complement of $\mathcal{D}_n$. Using arguments similar to those used in the proof of Theorem 2.1(i) and the proof of (A.5) in Gao *et al* (2009b), we can show that

$$\mathsf{P}\big(\mathcal{D}_n\big) = o(1). \tag{B.54}$$

Note that

$$\mathsf{P}\Big(\widehat{Q}_n(h) \leq x\Big) = \mathsf{P}\Big(\big\{\widehat{Q}_n(h) \leq x\big\} \cap \mathcal{D}_n^c\Big) + \mathsf{P}\Big(\big\{\widehat{Q}_n(h) \leq x\big\} \cap \mathcal{D}_n\Big).$$

This, together with (B.54), indicates

$$\mathsf{P}\Big(\widehat{Q}_n(h) \leq x\Big) = \mathsf{P}\Big(\big\{\widehat{Q}_n(h) \leq x\big\} \cap \mathcal{D}_n^c\Big) + o(1). \tag{B.55}$$

On the event $\mathcal{D}_n^c$, both $\frac{\overline{Q}_{n,2}(h)+\overline{Q}_{n,5}(h)}{\overline{\sigma}_n}$ and $\frac{\sigma_{n,1}-\overline{\sigma}_n}{\sigma_{n,1}}$ are bounded by s_n which converges to zero as n tends to infinity. Hence, by (B.53), (B.55) and (3.2), we have, as $n \to \infty$,

$$\Big|\mathsf{P}\left(\widehat{Q}_n(h) \leq x\right) - \Phi(x)\Big| \to 0 \tag{B.56}$$

uniformly in $x \in \mathcal{R}$. Similarly, we have, as $n \to \infty$,

$$\Big|\mathsf{P}^*\left(\widehat{Q}_n^*(h) \leq x\right) - \Phi(x)\Big| \to_P 0 \tag{B.57}$$

uniformly in $x \in \mathcal{R}$. We can then prove (3.5) by (B.56) and (B.57). The result (3.6) follows directly from (3.5). The proof of Proposition 3.1 is completed. ■

Proof of Proposition 3.2. Observe that

$$\begin{aligned}\beta_n^*(h) &= \mathsf{P}\left(\widehat{Q}_n(h) > l_\alpha^* | H_1^L\right) = 1 - \mathsf{P}\left(\widehat{Q}_n(h) \leq l_\alpha^* | H_1^L\right) \\ &= 1 - \mathsf{P}\Big(\widehat{Q}_{n,1}(h) \leq l_\alpha^* - \frac{\overline{Q}_{n,3}(h)}{\overline{\sigma}_n} - \frac{\sum_{j=2,\neq 3}^{6} \overline{Q}_{n,j}(h)}{\overline{\sigma}_n} | H_1^L\Big).\end{aligned} \tag{B.58}$$

Since the proof of Proposition 3.2(ii) is similar to that of Proposition 3.2(i), we only provide the proof of Proposition 3.2(i). Following the proof of Theorem 2.1, we have, as $n \to \infty$,

$$\overline{Q}_{n,i}(h) = o_P\left(nh\delta_n^2\right), \qquad i = 2, 4, 5, 6 \tag{B.59}$$

and

$$\mathsf{P}\left(c_3^* nh\delta_n^2 < \left|\overline{Q}_{n,3}(h)\right| < c_4^* nh\delta_n^2\right) \to 1, \tag{B.60}$$

for some $0 < c_3^* < c_4^* < \infty$. By (B.59) and (B.60), we may show that

$$\frac{\overline{Q}_{n,i}(h)}{\overline{\sigma}_n} = o_P\Big(\frac{\overline{Q}_{n,3}(h)}{\overline{\sigma}_n}\Big) \quad \text{for } i = 2, 4, 5, 6. \tag{B.61}$$

By (B.61), $\frac{\sum_{j=2,\neq 3}^{6} \overline{Q}_{n,j}(h)}{\overline{\sigma}_n}$ is dominated by $\frac{\overline{Q}_{n,3}(h)}{\overline{\sigma}_n}$ which is asymptotically equivalent to $\frac{\overline{Q}_{n,3}(h)}{\sigma_{n,1}}$ by (B.54). Hence, we have

$$\beta_n^*(h) \overset{P}{\sim} 1 - \mathsf{P}\Big(\widehat{Q}_{n,1}(h) \le l_\alpha^* - \frac{\overline{Q}_{n,3}(h)}{\sigma_{n,1}}(1 + o_P(1))\Big). \tag{B.62}$$

Noting that $l_\alpha^* - \frac{\overline{Q}_{n,3}(h)}{\sigma_{n,1}}$ is independent of $\{e_t\}$, by (3.2) and (B.62), we complete the proof of Proposition 3.2 (i). ■

Appendix C: Alternative estimation method of θ_0 and proofs of the technical lemmas

We start this appendix with the introduction of an alternative estimation method for θ_0 under H_1^L. The basic idea of the estimation method is to use the profile least squares approach, which involves the following two steps. Firstly, for given θ, estimate the distance function by a local linear smoothing method

$$\overline{\Delta}_n(v, \theta) = \sum_{t=1}^{n} \overline{w}_{nt}(v) \left[Y_t - g(V_t, \theta)\right],$$

where $\{\overline{w}_{nt}(v)\}$ is a sequence of weights given by

$$\overline{w}_{nt}(v) = \overline{L}_n\big(\frac{V_t - v}{b_1}\big) / \Big[\sum_{s=1}^{n} \overline{L}_n\big(\frac{V_s - v}{b_1}\big)\Big], \quad \overline{L}_n\big(\frac{V_t - v}{b_1}\big) = L\big(\frac{V_t - v}{b_1}\big)\left[\overline{S}_{n2}(v) - \big(\frac{V_t - v}{b_1}\big)\overline{S}_{n1}(v)\right],$$

$L(\cdot)$ is a kernel function, b_1 is a bandwidth and

$$\overline{S}_{nj}(v) = \frac{1}{\sqrt{n} b_1} \sum_{t=1}^{n} \big(\frac{V_t - v}{b_1}\big)^j L\big(\frac{V_t - v}{b_1}\big) \quad \text{for } j = 0, 1, 2.$$

Secondly, substitute $\overline{\Delta}_n(v,\theta)$ for $\Delta_n(v)$ and minimise the following nonlinear least squares with respect to θ to obtain the estimate of θ_0:

$$\begin{aligned}\widetilde{\theta} &= \arg\min_{\theta\in\Theta}\sum_{t=1}^{n}\left[Y_t - g(V_t,\theta) - \overline{\Delta}_n(V_t,\theta)\right]^2 \\ &= \arg\min_{\theta\in\Theta}\sum_{t=1}^{n}\left[\overline{Y}_t - \overline{g}(V_t,\theta)\right]^2, \end{aligned} \tag{C.1}$$

where $\overline{Y}_t = Y_t - \sum_{s=1}^{n}\overline{w}_{ns}(V_t)Y_s$ and $\overline{g}(V_t,\theta) = g(V_t,\theta) - \sum_{s=1}^{n}\overline{w}_{ns}(V_t)g(V_s,\theta)$. Following the standard argument for local linear smoothing (c.f., Fan and Gijbels, 1996), one may show that as $n\to\infty$,

$$\overline{g}(V_t,\theta) = c_g\ddot{g}_v(V_t,\theta)b_1^2\left[1+o_P(1)\right] \quad\text{and}\quad \widetilde{\Delta}_n(V_t) = c_\Delta\ddot{\Delta}_n(V_t)b_1^2\left[1+o_P(1)\right], \tag{C.2}$$

where $\ddot{g}_v(v,\theta) = \frac{d^2 g(v,\theta)}{dv^2}$, $\widetilde{\Delta}_n(V_t) = \Delta_n(V_t) - \sum_{s=1}^{n}\overline{w}_{ns}(V_t)\Delta_n(V_s)$, c_g and c_Δ are two constants and $\ddot{\Delta}_n(v) = \frac{d^2\Delta_n(v)}{dv^2}$. For derivatives of $g(v,\theta)$, we use the subscript v to indicate that the derivatives are taken with respect to v and the subscript θ to indicate they are taken with respect to θ. By some elementary calculation, one can also show that under the assumptions given in Section 2.3, the rate of convergence of the profile least squares estimator $\widetilde{\theta}$ is slower than that of the NLS estimator $\widehat{\theta}$. Hence, we propose using the NLS method to estimate θ_0. Chen, Gao, and Li (2011) studied the estimation of the time series model $Y_t = V_t\beta + \Delta(V_t) + e_t$, where $\{V_t, e_t\}$ is a stationary time series and β is a parameter. They pointed out that the conventional estimation method developed for the partially linear model $Y_t = V_t\beta + \Delta(U_t) + e_t$ with distinct regressors V_t and U_t in the linear and nonlinear components cannot be directly applied to the model with identical regressors in the two components, as otherwise the resulting estimator of β may be inconsistent.

We next provide the detailed proofs of the technical lemmas listed in Appendix A of the main article.

Proof of Lemma A.1. Recall that $p_t(\cdot)$ is the marginal density function of V_t and $p_{st}(\cdot,\cdot)$ is the joint density function of (V_t,V_s). Letting $q_t(\cdot)$ be the marginal density of $\frac{V_t}{\sqrt{t}}$, we may show that $p_t(v) = \frac{1}{\sqrt{t}}q_t\left(\frac{v}{\sqrt{t}}\right)$. Following the argument in Section 8.1 of the supplemental material Wang and Phillips (2012) and by using Assumption 1(i), we can show that $q_t(\cdot)$ is uniformly bounded. Hence, we can prove the first inequality in (A.1). Noting that

$$p_{st}(v,u) = \frac{1}{\sqrt{(t-s)s}}q_{st}\left(\frac{v-u}{\sqrt{t-s}},\frac{u}{\sqrt{s}}\right) = \frac{1}{\sqrt{(t-s)s}}q_{st}\left(\frac{v-u}{\sqrt{t-s}}\Big|\frac{u}{\sqrt{s}}\right)q_s\left(\frac{u}{\sqrt{s}}\right),$$

where $q_{st}(\cdot,\cdot)$ is the joint density function of $\left(\frac{V_t-V_s}{\sqrt{t-s}},\frac{V_s}{\sqrt{s}}\right)$ and $q_{st}(\cdot|u)$ is the conditional density of $\frac{V_t-V_s}{\sqrt{t-s}}$ given $\frac{V_s}{\sqrt{s}} = u$ for $t>s$, the second inequality in (A.1) follows similarly. ■

Proof of Lemma A.2(i). Define

$$\mathcal{L}_n(\theta) = \sum_{t=1}^{n} [Y_t - g(V_t, \theta)]^2$$

and

$$\dot{\mathcal{L}}_n(\theta) = 2\sum_{t=1}^{n} [Y_t - g(V_t, \theta)]\, \dot{g}_\theta(V_t, \theta).$$

Following the proof of Theorem 4.1 in Park and Phillips (2001), we may show that $\widehat{\theta}$ is a consistent estimator of θ_0. By a first-order Taylor expansion of $\dot{\mathcal{L}}_n(\theta)$, we have

$$0 = \dot{\mathcal{L}}_n(\widehat{\theta}) = \dot{\mathcal{L}}_n(\theta_0) + \ddot{\mathcal{L}}_n(\theta_\diamond)(\widehat{\theta} - \theta_0), \tag{C.3}$$

where $\theta_\diamond$ lies in the line segment connecting $\widehat{\theta}$ and θ_0, and

$$\begin{aligned} \ddot{\mathcal{L}}_n(\theta) &= -2\sum_{t=1}^{n} \dot{g}_\theta(V_t, \theta)\big[\dot{g}_\theta(V_t, \theta)\big]^T + 2\sum_{t=1}^{n} (Y_t - g(V_t, \theta))\, \ddot{g}_\theta(V_t, \theta) \\ &\equiv \ddot{\mathcal{L}}_{n1}(\theta) + \ddot{\mathcal{L}}_{n2}(\theta) \end{aligned}$$

with $\ddot{g}_\theta(v, \theta)$ being the second derivative of $g(v, \theta)$ with respect to θ.

By Theorem 3.2 in Park and Phillips (2001), we have, uniformly for $\theta \in \Theta$,

$$\frac{1}{\sqrt{n}}\ddot{\mathcal{L}}_{n1}(\theta) \to_P -2\left(\int_{-\infty}^{\infty} \dot{g}_\theta(v, \theta)\big[\dot{g}_\theta(v, \theta)\big]^T dv\right) L_B(1, 0), \tag{C.4}$$

where $L_B(1, 0)$ is the local time process of the standard Brownian motion $B(t)$ at point 0 over the time interval $[0, 1]$. On the other hand, under H_1^L,

$$\begin{aligned} \ddot{\mathcal{L}}_{n2}(\theta_\diamond) &= 2\sum_{t=1}^{n} e_t \ddot{g}_\theta(V_t, \theta_\diamond) + 2\sum_{t=1}^{n} \Delta_n(V_t)\ddot{g}_\theta(V_t, \theta_\diamond) + 2\sum_{t=1}^{n} \big[g(V_t, \theta_0) - g(V_t, \theta_\diamond)\big]\ddot{g}_\theta(V_t, \theta_\diamond) \\ &\equiv 2\ddot{\mathcal{L}}_{n3}(\theta_\diamond) + 2\ddot{\mathcal{L}}_{n4}(\theta_\diamond) + 2\ddot{\mathcal{L}}_{n5}(\theta_\diamond). \end{aligned} \tag{C.5}$$

By Definition 2.5 in Section 2.2 of the main article, there exists an integrable function $\Gamma(\cdot)$ such that

$$\Delta_n^2(v) \le \delta_n^2 \Gamma(v).$$

By the Cauchy-Schwarz inequality and Theorem 5.1 in Park and Phillips (1999), we have

$$\begin{aligned} \|\ddot{\mathcal{L}}_{n4}(\theta_\diamond)\| &= \left\|\sum_{t=1}^{n} \Delta_n(V_t)\ddot{g}_\theta(V_t, \theta_\diamond)\right\| \\ &\le \left(\sum_{t=1}^{n} \Delta_n^2(V_t)\right)^{1/2} \left(\sum_{t=1}^{n} \|\ddot{g}_\theta(V_t, \theta_\diamond)\|^2\right)^{1/2} \\ &\le \delta_n \left(\sum_{t=1}^{n} \Gamma(V_t)\right)^{1/2} \left(\sum_{t=1}^{n} \|\ddot{g}_\theta(V_t, \theta_\diamond)\|^2\right)^{1/2} \\ &= O_P(\sqrt{n}\delta_n) = o_P(\sqrt{n}) \end{aligned} \tag{C.6}$$

as $\delta_n \to 0$. Following the proof of Theorem 5.1 in Park and Phillips (2001) and noting that $\left|\theta_\diamond - \theta_0\right| = o_P(1)$, we have

$$\|\ddot{\mathcal{L}}_{n3}(\theta_\diamond)\| = O_P(\sqrt[4]{n}) = o_P(\sqrt{n}), \quad \|\ddot{\mathcal{L}}_{n5}(\theta_\diamond)\| = o_P(\sqrt{n}). \tag{C.7}$$

By (C.4)–(C.7), we have

$$\begin{aligned} \frac{1}{\sqrt{n}}\ddot{\mathcal{L}}_n(\theta_\diamond) &= \frac{1}{\sqrt{n}}\ddot{\mathcal{L}}_{n1}(\theta_\diamond) + o_P(1) \\ &\to_P -2\left(\int_{-\infty}^{\infty} \dot{g}_\theta(v,\theta_\diamond)\left[\dot{g}_\theta(v,\theta_\diamond)\right]^T dv\right) L_B(1,0). \end{aligned} \tag{C.8}$$

Since $\widehat{\theta}$ is a consistent estimator of θ_0, we have $\theta_\diamond - \theta_0 = o_P(1)$. Hence, $\ddot{\mathcal{L}}_n(\theta_\diamond) = \ddot{\mathcal{L}}_n(\theta_0)(1 + o_P(1))$. In view of (C.3), we have

$$0 = \dot{\mathcal{L}}_n(\widehat{\theta}) \overset{P}{\sim} \dot{\mathcal{L}}_n(\theta_0) + \ddot{\mathcal{L}}_n(\theta_0)(\widehat{\theta} - \theta_0).$$

By the Convexity Lemma in Pollard (1991), we have

$$\widehat{\theta} - \theta_0 \overset{P}{\sim} -\ddot{\mathcal{L}}_n^{-1}(\theta_0)\dot{\mathcal{L}}_n(\theta_0). \tag{C.9}$$

Under the sequence of local alternatives H_1^L, we have

$$\dot{\mathcal{L}}_n(\theta_0) = 2\sum_{t=1}^n \dot{g}_\theta(V_t,\theta_0)e_t + 2\sum_{t=1}^n \dot{g}_\theta(V_t,\theta_0)\Delta_n(V_t). \tag{C.10}$$

Following the proof of Theorem 5.1 in Park and Phillips (2001) again, we have

$$\sum_{t=1}^n \dot{g}_\theta(V_t,\theta_0)e_t = O_P(\sqrt[4]{n}). \tag{C.11}$$

By the Cauchy-Schwarz inequality and Theorem 5.1 in Park and Phillips (1999), we have

$$\begin{aligned} \Big\|\sum_{t=1}^n \dot{g}_\theta(V_t,\theta_0)\Delta_n(V_t)\Big\| &\le \Big(\sum_{t=1}^n \big\|\dot{g}_\theta(V_t,\theta_0)\big\|^2\Big)^{1/2}\Big(\sum_{t=1}^n \Delta_n^2(V_t)\Big)^{1/2} \\ &= O_P(\sqrt[4]{n}\delta_n)\Big(\sum_{t=1}^n \Gamma(V_t)\Big)^{1/2} = O_P(\sqrt{n}\delta_n). \end{aligned} \tag{C.12}$$

In view of (C.8)–(C.12), we can show that Lemma A.2(i) holds. ■

Proof of Lemma A.2(ii). When Assumption 3′ is satisfied, by Theorem 3.3 in Park and Phillips (2001), we have

$$\begin{aligned} \frac{1}{n\dot{\kappa}^2(\sqrt{n})}\ddot{\mathcal{L}}_{n1}(\theta_0) &= -\frac{2}{n\dot{\kappa}^2(\sqrt{n})}\sum_{t=1}^n \dot{g}_\theta(V_t,\theta_0)\left[\dot{g}_\theta(V_t,\theta_0)\right]^T \\ &\to_P -2\int_0^1 \dot{h}(B(r),\theta)\left[\dot{h}(B(r),\theta)\right]^T dr \neq 0. \end{aligned} \tag{C.13}$$

By Definition 2.6, there exists an asymptotically homogeneous function $\Lambda(\cdot)$ with order $v^2(\cdot)$ satisfying $\Delta_n^2(v) \leq \delta_n^2\Lambda(v)$. Hence, by Theorem 5.2 in Park and Phillips (2001) and following the proof of Lemma A.2(i), we can complete the proof of Lemma A.2(ii). ■

Proof of Lemma A.3. Lemma A.3 can be proved by using Theorem 2.1 in Wang and Phillips (2009) and details are omitted here. ■

Proof of Lemma A.4(i). In order to prove Lemma A.4(i), it suffices to evaluate the order of $\sum_{t=1}^n\sum_{s=1,\neq t}^n K_{s,t}^2$. Following the proof of (B.35), we have

$$\begin{aligned}\sum_{t=1}^n\sum_{s=1,\neq t}^n K_{s,t}^2 &= \sum_{t=1}^n\sum_{s=1}^n K_{s,t}^2 - nK^2(0) \geq \sum_{t=1}^{[\frac{n}{3}]}\sum_{s=[\frac{n}{2}]}^n K_{s,t}^2 - nK^2(0) \\ &\geq c_1^{\diamond}\sum_{t=1}^{[\frac{n}{3}]}\sqrt{n}h - nK^2(0) \geq c_1 n^{3/2}h \qquad \text{(C.14)}\end{aligned}$$

in probability, where $0 < c_1 < c_1^{\diamond} < \infty$. Similarly, we can also show that

$$\sum_{t=1}^n\sum_{s=1,\neq t}^n K_{s,t}^2 \leq c_2 n^{3/2}h \qquad \text{(C.15)}$$

in probability, where $0 < c_1 < c_2 < \infty$. By (C.14) and (C.15), we complete the proof of Lemma A.4(i). ■

Proof of Lemma A.4(ii). Let

$$\eta_t = \sum_{s=1}^{t-1} K_{t,s}e_s, \quad q_n^2 = \sigma_e^4\sum_{t=2}^n\sum_{s=1}^{t-1} K_{t,s}^2, \quad U_{nt} = (\eta_t e_t)/q_n$$

and $\Omega_{n,t} = \sigma(e_1, \cdots, e_t; V_1, \cdots, V_n)$ be the σ-field generated by $\{(e_i, V_j) : 1 \leq i \leq t; 1 \leq j \leq n\}$. Note that

$$\Omega_{n,t} = \sigma(e_1, \cdots, e_t; V_1, \cdots, V_n) \subseteq \mathcal{F}_{n,t} = \sigma(e_1, \cdots, e_t; \varepsilon_{-\infty}, \cdots, \varepsilon_{t+1}; \varepsilon_{t+2}, \cdots, \varepsilon_n).$$

Recall that $\mathcal{F}_t = \sigma(e_1, \cdots, e_t; \varepsilon_{-\infty}, \cdots, \varepsilon_{t+1})$ as defined in Assumption 1(ii) and ε_t is independent of e_s, $t \geq s+1$. Then, we have

$$\begin{aligned}&\mathsf{E}[e_t|\mathcal{F}_{n,t-1}] = \mathsf{E}[e_t|\sigma(e_1, \cdots, e_{t-1}; \varepsilon_{-\infty}, \cdots, \varepsilon_t; \varepsilon_{t+1}, \cdots, \varepsilon_n)] \\ &= \mathsf{E}[e_t|\sigma(e_1, \cdots, e_{t-1}; \varepsilon_{-\infty}, \cdots, \varepsilon_t)] = \mathsf{E}[e_t|\mathcal{F}_{t-1}] = 0 \ \ a.s.,\end{aligned}$$

where we have used the fact that $(e_t, e_{t-1}, \cdots, e_1; \varepsilon_t, \varepsilon_{t-1}, \cdots, \varepsilon_{-\infty})$ and $(\varepsilon_{t+1}, \cdots, \varepsilon_n)$ are independent. Hence, $\{(U_{nt}, \Omega_{n,t})\}$ forms a sequence of martingale differences.

By Theorem 1 of Pollard (1984, p. 171) or Corollary 3.1 of Hall and Heyde (1980), to prove

$$\sum_{t=2}^{n} U_{nt} \to_D \mathsf{N}(0,1), \tag{C.16}$$

it suffices to show that, for any $\delta > 0$,

$$\sum_{t=2}^{n} \mathsf{E}\left[U_{nt}^2 I_{\{[U_{nt}|>\delta]\}}|\Omega_{n,t-1}\right] \quad \to_P \quad 0, \tag{C.17}$$

$$\sum_{t=2}^{n} \mathsf{E}\left[U_{nt}^2|\Omega_{n,t-1}\right] \quad \to_P \quad 1. \tag{C.18}$$

In view of the definition of $\{U_{nt}\}$, in order to prove (C.17) and (C.18), it suffices to show that, as $n \to \infty$,

$$\frac{1}{q_n^4}\sum_{t=2}^{n} \eta_t^4 \quad \to_P \quad 0 \qquad \text{and} \tag{C.19}$$

$$\frac{\sigma_e^2}{q_n^2}\sum_{t=2}^{n} \eta_t^2 \quad \to_P \quad 1. \tag{C.20}$$

The proofs of (C.19) and (C.20) are given in Lemmas C.1 and C.2 below, respectively.

Lemma C.1. *Under the conditions of Lemma A.4(ii), we have, as $n \to \infty$,*

$$\frac{\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1} K_{s,t}e_s\right)^4}{\left(\sum_{t=2}^{n}\sum_{s=1}^{t-1} K_{s,t}^2\right)^2} \to_P 0. \tag{C.21}$$

Proof. Observe that

$$\mathsf{E}\left[\eta_t^4\right] = \sum_{s_1=1}^{t-1}\sum_{s_2=1}^{t-1}\sum_{s_3=1}^{t-1}\sum_{s_4=1}^{t-1} \mathsf{E}\left[K_{s_1,t}K_{s_2,t}K_{s_3,t}K_{s_4,t}e_{s_1}e_{s_2}e_{s_3}e_{s_4}\right]. \tag{C.22}$$

By Assumption 1(ii), we have, for all $1 < s < t$,

$$\begin{aligned} & \mathsf{E}\left[e_s|\sigma(e_{s-1},\cdots,e_1;\varepsilon_s,\cdots,\varepsilon_{-\infty};\varepsilon_t,\cdots,\varepsilon_{s+1})\right] \\ = \quad & \mathsf{E}\left[e_s|\sigma(e_{s-1},\cdots,e_1;\varepsilon_s,\cdots,\varepsilon_{-\infty})\right] = 0 \quad a.s. \end{aligned} \tag{C.23}$$

Thus, by equation (C.23), we have, for all $1 < s < t$,

$$\mathsf{E}\left[e_s|\sigma\left(e_{s-1},\cdots,e_1;v_t,\cdots,v_1\right)\right] = 0 \ a.s. \tag{C.24}$$

There are several terms involved in a sum of the form: $\sum_{s_1=1}^{t-1}\sum_{s_2=1}^{t-1}\sum_{s_3=1}^{t-1}\sum_{s_4=1}^{t-1}$. We start with the case that s_1, s_2, s_3 and s_4 are all different. Without loss of generality, let

$1 \le s_4 < s_3 < s_2 < s_1 < t$ in the following discussion. By (C.24), we have, for $1 \le s_4 < s_3 < s_2 < s_1 < t$,

$$\begin{aligned} & \mathsf{E}\left[e_{s_1}e_{s_2}e_{s_3}e_{s_4}|\sigma\left(v_{s_4+1},\cdots,v_{s_3};v_{s_3+1},\cdots,v_{s_2};v_{s_2+1},\cdots,v_{s_1};v_{s_1+1},\cdots,v_t;e_{s_4},e_{s_3},e_{s_2}\right)\right] \\ = \; & e_{s_2}e_{s_3}e_{s_4}\mathsf{E}\left[e_{s_1}|\sigma\left(v_{s_4+1},\cdots,v_{s_3};v_{s_3+1},\cdots,v_{s_2};v_{s_2+1},\cdots,v_t;e_{s_4},e_{s_3},e_{s_2}\right)\right] \\ = \; & 0 \;\; a.s. \end{aligned} \tag{C.25}$$

Let $\mathbb{K}(s_1,\cdots,s_4,t) = K_{s_1,t}K_{s_2,t}K_{s_3,t}K_{s_4,t}$. Equation (C.25) implies that, for $1 \le s_4 < s_3 < s_2 < s_1 < t$,

$$\begin{aligned} & \mathsf{E}\big[\mathbb{K}(s_1,\cdots,s_4,t)e_{s_1}e_{s_2}e_{s_3}e_{s_4}\big|\sigma\left(v_{s_4+1},\cdots,v_{s_3};v_{s_3+1},\cdots,v_{s_2};v_{s_2+1},\cdots,v_{s_1};v_{s_1+1},\cdots,v_t\right)\big] \\ = \; & \mathbb{K}(s_1,\cdots,s_4,t)\cdot\mathsf{E}\big[e_{s_1}e_{s_2}e_{s_3}e_{s_4}\big|\sigma\left(v_{s_4+1},\cdots,v_{s_3};v_{s_3+1},\cdots,v_{s_2};v_{s_2+1},\cdots,v_{s_1};v_{s_1+1},\cdots,v_t\right)\big] \\ = \; & 0 \;\; a.s., \end{aligned} \tag{C.26}$$

which implies that

$$\mathsf{E}\big[\mathbb{K}(s_1,\cdots,s_4,t)e_{s_1}e_{s_2}e_{s_3}e_{s_4}\big] = 0 \tag{C.27}$$

for $1 \le s_4 < s_3 < s_2 < s_1 < t$. Similarly, for the case that at least three of $s_1,\ s_2,\ s_3,\ s_4$ are different, equation (C.27) also holds. Hence, by (C.14) and (C.15), in order to prove (C.21), it suffices to show that, as $n \to \infty$,

$$\frac{1}{n^3h^2}\sum_{t=2}^{n}\sum_{s_1=1}^{t-1}\sum_{s_2=1,\neq s_1}^{t-1}\mathsf{E}\left[K_{t,s_1}^2K_{t,s_2}^2e_{s_1}^2e_{s_2}^2\right] \;\to\; 0, \tag{C.28}$$

$$\frac{1}{n^3h^2}\sum_{t=2}^{n}\sum_{s=1}^{t-1}\mathsf{E}\big[K_{t,s}^4e_s^4\big] \;\to\; 0. \tag{C.29}$$

We first prove (C.28). By the same arguments as that in the proof of (B.16) in Appendix B and Assumption 1(ii), we have, for $1 \le s_2 < s_2 < t$,

$$\mathsf{E}\left[K_{t,s_1}^2K_{t,s_2}^2\ e_{s_1}^2e_{s_2}^2\right] = \sigma_e^4\ \mathsf{E}\left[K_{t,s_1}^2K_{t,s_2}^2\right]. \tag{C.30}$$

Let $f_{st}(\cdot)$ be the density function of $V_t - V_s$ for $t > s$ and $g_{st}(\cdot)$ be the density function of $\frac{1}{\sqrt{t-s}}(V_t - V_s) = \frac{1}{\sqrt{t-s}}\sum_{i=s+1}^{t} v_i$. By Lemma A.1, we have, as $t - s \to \infty$,

$$f_{st}(x) = \frac{1}{\sqrt{t-s}}g_{st}\Big(\frac{1}{\sqrt{t-s}}\Big) \le \frac{c_0}{\sqrt{t-s}} \tag{C.31}$$

uniformly for x, where c_0 is sufficiently large. Following similar arguments to those in the proof of Lemma A.2 in Gao *et al* (2009b), we have

$$
\begin{aligned}
&\sum_{t=2}^{n}\sum_{s_1=1}^{t-1}\sum_{s_2=1,\neq s_1}^{t-1} \mathsf{E}\left[K^2\left(\frac{V_t-V_{s_1}}{h}\right)K^2\left(\frac{V_t-V_{s_2}}{h}\right)\right] \\
&= 2\sum_{t=3}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1} \mathsf{E}\left[K^2\left(\frac{V_t-V_{s_1}}{h}\right)K^2\left(\frac{V_t-V_{s_1}}{h}+\frac{V_{s_1}-V_{s_2}}{h}\right)\right] \\
&\leq 2c_0^2h^2\,(1+o(1))\left(\int_{-\infty}^{\infty}K^2(u)du\right)^2\sum_{t=3}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}\frac{1}{\sqrt{t-s_1}}\frac{1}{\sqrt{s_1-s_2}} \\
&= O(n^2h^2) = o\left(n^3h^2\right), \qquad (C.32)
\end{aligned}
$$

where we have used the convergence result in (C.31). Then (C.28) follows from (C.32). In a very similar fashion, we can prove (C.29). This completes the proof of Lemma C.1. ■

LEMMA C.2. *Under the conditions of Lemma A.4(ii), we have, as $n\to\infty$,*

$$\frac{1}{\sum_{t=2}^{n}\sum_{s=1}^{t-1}K_{s,t}^2}\sum_{t=2}^{n}\left(\eta_t^2-\sum_{s=1}^{t-1}K_{s,t}^2\sigma_e^2\right)\to_P 0. \qquad (C.33)$$

PROOF. Observe that

$$\sum_{t=2}^{n}\eta_t^2=\sum_{t=2}^{n}\big(\sum_{s=1}^{t-1}K_{s,t}e_s\big)^2=\sum_{t=2}^{n}\sum_{s=1}^{t-1}K_{s,t}^2e_s^2+2\sum_{t=3}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}e_{s_1}K_{s_1,t}K_{s_2,t}e_{s_2}. \qquad (C.34)$$

We first show that, as $n\to\infty$,

$$\frac{1}{\sum_{t=2}^{n}\sum_{s=1}^{t-1}K_{s,t}^2}\sum_{t=2}^{n}\sum_{s=1}^{t-1}K_{s,t}^2\left(e_s^2-\sigma_e^2\right)\to_P 0. \qquad (C.35)$$

Similarly to the proof of Lemma C.1, it can be shown that

$$\sum_{t=2}^{n}\sum_{s=1}^{t-1}K_{s,t}^2\left(e_s^2-\sigma_e^2\right) = o_P(n^{3/2}h), \qquad (C.36)$$

which leads to (C.35). In view of (C.34) and (C.35), to prove (C.33), we need to show that, as $n\to\infty$,

$$\frac{1}{n^{3/2}h}\sum_{t=2}^{n}\sum_{s_1=1}^{t-1}\sum_{s_2=1,\neq s_1}^{t-1} e_{s_1}\;K_{s_1,t}K_{s_2,t}\;e_{s_2}\to_P 0. \qquad (C.37)$$

It suffices to show that, as $n\to\infty$,

$$\frac{1}{n^3h^2}\mathsf{E}\big[\sum_{t=3}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}K_{s_1t}K_{s_2t}e_{s_2}e_{s_1}\big]^2\to 0. \qquad (C.38)$$

Note that

$$
\begin{aligned}
&\mathsf{E}\Big[\sum_{t=3}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1} K_{s_1t}K_{s_2t}\ e_{s_2}e_{s_1}\Big]^2 \\
= &\sum_{t_1=3}^{n}\sum_{t_2=3}^{n}\sum_{s_1=2}^{t_1-1}\sum_{s_2=2}^{t_2-1}\sum_{s_3=1}^{s_1-1}\sum_{s_4=1}^{s_2-1}\mathsf{E}\big[K_{t_1,s_1}K_{t_1,s_3}K_{t_2,s_2}K_{t_2,s_4}e_{s_1}e_{s_2}e_{s_3}e_{s_4}\big] \qquad \text{(C.39)}\\
= &\ \sigma_e^4\sum_{t_1=3}^{n}\sum_{t_2=3}^{n}\sum_{s_1=2}^{t_1-1}\sum_{s_2=2}^{t_2-1}\mathsf{E}\big[K_{t_1,s_1}^2K_{t_2,s_2}^2\big]+2\sigma_e^4\sum_{t_1=3}^{n}\sum_{t_2=3}^{n}\sum_{s_1=2}^{t_1-1}\sum_{s_2=2}^{t_2-1}\mathsf{E}\big[K_{t_1,s_1}K_{t_1,s_3}K_{t_2,s_2}K_{t_2,s_4}\big].
\end{aligned}
$$

In order to evaluate the order of (C.39), we only consider the case for $1 \le s_2 < s_1 < t_2 < t_1 \le n$ as the evaluation in other cases is similar. When $1 \le s_2 < s_1 < t_2 < t_1 \le n$, similarly to the derivations in (C.32), we can show that the last term in (C.39) becomes

$$
\begin{aligned}
&2\sigma_e^4\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}\mathsf{E}\Big[K\big(\frac{V_{t_1}-V_{t_2}}{h}+\frac{V_{t_2}-V_{s_1}}{h}\big)K\big(\frac{V_{t_1}-V_{t_2}}{h}+\frac{V_{t_2}-V_{s_1}}{h}+\frac{V_{s_1}-V_{s_2}}{h}\big)\\
&\qquad\qquad \times K\big(\frac{V_{t_2}-V_{s_1}}{h}\big)K\big(\frac{V_{t_2}-V_{s_1}}{h}+\frac{V_{s_1}-V_{s_2}}{h}\big)\Big]\\
\le &\ Ch^3\sum_{t_1=4}^{n}\sum_{t_2=3}^{t_1-1}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}\frac{1}{\sqrt{t_1-t_2}}\,\frac{1}{\sqrt{t_2-s_1}}\,\frac{1}{\sqrt{s_1-s_2}}\\
= &\ O(n^{\frac{5}{2}}h^3)=o(n^3h^2), \qquad \text{(C.40)}
\end{aligned}
$$

where $C > 0$ is sufficiently large. We can also show that the first term on the right-hand side of the last equality in (C.39) is of a smaller order than n^3h^2. Thus, the proof of (C.38) can be completed. This completes the proof of Lemma C.2 and therefore Lemma A.4(ii). ■

We next present a lemma established by Götze *et al* (2007) which has played a crucial role in the proof of Proposition 3.1. Note that the assumption of continuous randomness of $\{X_i\}$ is needed for the establishment of the Edgeworth expansion for the quadratic form in Lemma C.3 below. Under the continuous randomness of $\{X_i\}$, the Cramér's condition is satisfied automatically (see p.45 of Hall 1992).

Consider the following quadratic form

$$
Q_n = \sum_{j=1}^{n} a_{jj}(X_j^2 - \mathsf{E}[X_j^2]) + \sum_{j\neq k} X_j a_{jk} X_k, \qquad \text{(C.41)}
$$

where $\{X_j\}$ is a sequence of i.i.d. continuous random variables with $\mathsf{E}[X_1] = 0$, $\mathsf{E}[X_1^2] = 1$ and $\mathsf{E}[|X_1|^6] < \infty$, $\{a_{jk}\}$ is a sequence of real numbers possibly depending on n. Let $\mathbf{A} = (a_{jk})_{j,k=1}^n$ be an $n \times n$ matrix with entries a_{jk}, and define $\|\mathbf{A}\| = \sum_{j,k=1}^n a_{jk}^2$ and $\mathsf{Tr}(\mathbf{A}) = \sum_{j=1}^n a_{jj}$, and λ_1 as the maximal eigenvalue of $\mathbf{A}$ in absolute value. Further define

$$
V^2 = \sum_{j=1}^{n} a_{jj}^2, \quad \mathcal{L}_j^2 = \sum_{k=1}^{n} a_{jk}^2, \ j = 1, \cdots, n, \quad \mathbf{d} = (a_{11}, a_{22}, \cdots, a_{nn})',
$$

and $\mu_k = \mathsf{E}[X_1^k]$, $\beta_k = \mathsf{E}[|X_1|^k]$, $k = 1, \cdots, 6$. Let $\mathbf{A}_0$ denote an $n \times n$ matrix with zero diagonal entries and define

$$M_n = \max\left\{ \frac{|\lambda_1|^2}{\|\mathbf{A}\|^2}, \; \frac{\left(\sum_{j=1}^n \mathcal{L}_j^4\right)^{1/2}}{\|\mathbf{A}\|^2} \right\},$$

$$\sigma_*^2 = (\mu_4 - \mu_2^2)V^2 + 2\mu_2^2\|\mathbf{A_0}\|^2, \qquad \kappa = \sigma_*^{-3}\left(\mu_3^2\mathbf{d}'\mathbf{A_0}\mathbf{d} + \frac{4}{3}\mu_2^3\mathsf{Tr}(\mathbf{A_0^3})\right).$$

The coefficients in the quadratic form (C.41) should satisfy the following conditions.

Q (i) $\|\mathbf{A}\| < \infty$.

Q (ii) There exists some absolute positive constant $b_1^2 > 0$ such that

$$1 - \frac{V^2}{\|\mathbf{A}\|^2} \geq b_1^2.$$

Lemma C.3 below establishes an Edgeworth expansion for the quadratic form defined in (C.41). A detailed proof is available in Götze *et al* (2007).

LEMMA C.3. *Under conditions Q(i) and Q(ii), we have*

$$\sup_{x\in\mathcal{R}}\left|\mathsf{P}(Q_n/\sigma_* \leq x) - \Phi(x) + \kappa\Phi^{(3)}(x)\right| \leq C_1 b_1^{-4}\left(\beta_3^2 + V\|\mathbf{A}\|^{-1}\beta_6\right)\mu_2^{-3}M_n,$$

where C_1 is some positive constant.

In addition to Lemma C.3, we also need Lemma C.4 below to prove Proposition 3.1. It follows from Theorem 2.1 of Wang and Phillips (2009), and a rigorous proof is given in Wang and Chan (2012)[1]. A similar result is also given in Theorem 4.1 of Gao *et al* (2014).

LEMMA C.4. *Under Assumptions 1(i) and 2, we have, as $n \to \infty$,*

$$P\left(\sup_{|v|\leq C_2\sqrt{n}}\left|\frac{1}{\sqrt{n}h}\sum_{t=1}^n K\left(\frac{V_t - v}{h}\right)\right| \geq C_2^*\right) \leq \varepsilon \tag{C.42}$$

for some small $0 < \varepsilon < 1$ and $0 < C_2, C_2^* < \infty$.

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[1]The authors thank Dr Qiying Wang for making their working paper available for it being cited in this supplementary document.

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